DOLLAR COST AVERAGING VS OPTIMAL BUY-AND-HOLD IN A MODEL WITH EQUITY MOMENTUM, REVERSION, AND MISPRICING

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Abstract

In a fundamental dynamic allocation model where the equity price process has both momentum and reversion, reflecting transient mispricing, scenarios are found where typical risk-averse investors prefer dollar cost averaging to plunging a lump sum into an optimal buy-and-hold allocation.

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In this study's fundamental model of dynamic allocation between risky and risk-free assets, many investors would be better off to use dollar cost averaging (DCA), accumulating the risky asset gradually by installments, instead of plunging lump sum funds into an optimal buy-and-hold allocation. This theoretical finding is in stark contrast to the well-known result in the traditional random walk model of Merton (1969) and Samuelson (1969), where DCA is obviously suboptimal because any investor's optimal mix of risky and risk-free assets is constant and independent of the investment horizon.

In our model, the price of the risky asset (equity) may deviate temporarily from fair value, which underlies an assumed price process with both momentum and reversion instead of a random walk. The process is fashioned from modern empirical findings, especially Poterba and Summers (1988), who conclude that empirically-observed equity momentum over short periods and reversion patterns over longer periods reflect transient equity mispricing.

The Merton-Samuelson random walk model assumes that investors rebalance each period to their constant optimal allocation. In contrast, our model's complex equity price process renders an optimal reallocation plan that a typical investor would find too difficult to determine or implement. Therefore, our model's investors are reasonably assumed to choose between DCA and plunging a lump sum into the optimal buy-and-hold allocation.

Other than the price process and reallocation assumptions, our model is intentionally similar to the Merton-Samuelson model: (1) the risk-free rate is constant; and (2) investors have constant relative risk aversion, where allocation decisions are independent of wealth. The question is whether DCA is a viable investment strategy in the model. The results indicate that with sufficiently strong momentum and reversion tendencies, the answer is yes.

BACKGROUND

Financial advisors often advocate DCA, arguing that the strategy avoids plunging a lump sum into equity at what could turn out to be "the wrong time" and reduces risk by diversifying the equity purchase price over time. After DCA was exposed as obviously suboptimal in the Merton-Samuelson model, academic literature further condemned DCA (e.g., Constantinides, 1979; Knight and Mandell, 1993; Rozeff, 1994; and Thorley, 1994).

Despite the negative academic opinions, financial advisors persisted in recommending DCA. Subsequent academic efforts have largely searched for scenarios to explain that persistence. Some of the research is behavioral finance (e.g., Statman, 1995; and Dichtl and Drobetz, 2011), and some is more traditional finance (e.g., Balvers and Mitchell, 2000; Milevsky and Posner, 2003; Brennan et al., 2005; Cho and Kuvvet, 2015; and Smith and Artigue, 2018). The traditional finance studies reach differing conclusions with different risk metrics and different historical data estimates. Therefore, at this time, there is no consensus in the literature on what may be called the "DCA puzzle".[1]

[1] A broad overview of the DCA literature is in Smith and Artigue (2018). Brennan et al. (2005) provide a selective review with more depth.

Many financial advisors also advocate a higher equity allocation for a longer investment horizon, arguing that a longer horizon allows for more "time diversification" of portfolio wealth outcomes. This strategy implies an equity allocation percentage that tends to decrease over time as the investment horizon gets nearer, which conflicts with DCA where the equity allocation percentage increases over time. Both financial advisor strategies are in conflict with the Merton-Samuelson random walk model.

Instead of a random walk, Samuelson's (1991) model considers reversion and momentum processes (separately), with results that reconcile somewhat with both advisor strategies. If equity prices follow a reversion process, investors with typical risk aversion levels will optimally allocate more to equity for longer horizons, regardless of whether the investor is a buy-and-holder or optimally reallocates each period. Because empirical evidence supports equity price reversion (e.g., Campbell et al., 2001; Gropp, 2004; Bali et al., 2008; and Pástor and Stambaugh, 2012), Samuelson's (1991) result vindicates the advisor "time diversification" strategy of a higher equity allocation for a longer investment horizon. Kritzman (1994, 2015) illustrates this horizon effect for buy-and-holders.[2]

Because a momentum process is the opposite of a reversion process, the Samuelson (1991) model implies that with momentum, investors with typical risk aversion levels will optimally allocate less to equity for longer horizons. Because this horizon effect implies a tendency for an investor's equity allocation to gradually increase over time, the investment strategy bears a

[2] Based on an empirical analysis supporting an equity price reversal process (mean reversion) instead of a random walk, Lee (1990) also defended the "time diversification" strategy. Theoretical and empirical researchers have elaborated on this horizon effect (e.g., Kim and Omberg, 1996; Campbell and Viceira, 1999; Barberis, 2000; Campbell et al., 2001; and Wachter, 2002).

resemblance to DCA. Empirical evidence supports momentum in equity prices (e.g., Jagadeesh and Titman, 1993; Carhart, 1997; Asness et al., 2014; and Subrahmanyam, 2018).

Because momentum and reversion processes are opposites, empirical support for both may seem contradictory. However, the reconciliation is that momentum is observed in returns over short intervals, whereas reversion is observed in returns for longer intervals (e.g., Fama and French, 1988; Poterba and Summers, 1988; and Balvers and Wu, 2006).

The goal of the present study is to explore the viability of DCA in a model with equity momentum, reversion, and mispricing. Our model is adapted from Kritzman's (1994) simple binomial model of the "time diversification" issue, in hopes of providing an a similarly insightful and instructive analysis. An empirical analysis would be useful, but is beyond the scope of the present study. We hope that empiricists will consider this potential research opportunity.

EQUITY PRICE PROCESS

Equity prices are assumed to follow a 3-year binomial process with six (6) half-year periods. The beginning equity price is normalized to $1, and each up-move (down-move) is 1.15 (0.94) times the prior period's realized equity price (dividends reinvested). When the single-period up- and down-move probabilities are each 0.50, equity's half-year expected rate of return is 4.5%, or roughly 9% annually. Fixed income securities are assumed to yield a constant risk-free rate of 2% per half year, or roughly 4% annually. Thus, the half-year average equity risk premium in this case is 2.5%, or roughly 5% annually.[3]

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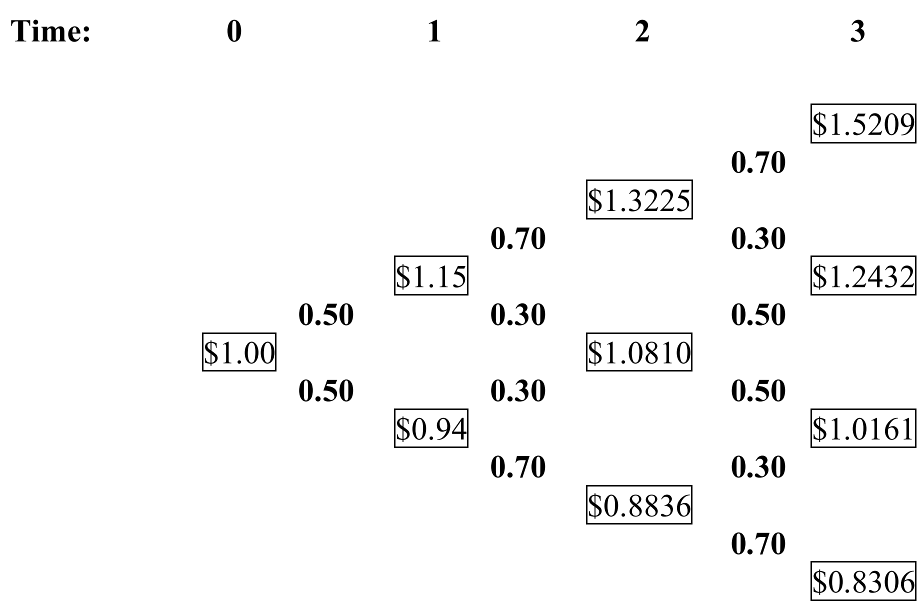
[3] A 4% annual risk-free rate is not too different from the long-run average 3-month U.S. Treasury rate (Damodaran Online: Home Page for Aswath Damodaran (nyu.edu).) An annual equity risk premium of 5% is in line with the *ex ante* and historical U.S. equity market estimates (same site).

Momentum Process

Over the first three periods, equity price is assumed to follow the simple momentum process shown in Exhibit 1. In the momentum process, the probability of an up-move in the top half of the binomial tree is assumed to be higher after an up-move. The bottom half of the tree has the mirror structure for down-moves. At time 0 and in the middle of the tree at time 2, the probability of an up(down)-move is 0.50 (0.50). Exhibit 1 shows the assumed up- and down-move momentum process probabilities in boldface.

EXHIBIT 1: EQUITY MOMENTUM PROCESS



Beginning at time 3, equity price is assumed to follow a reversion process. To specify the reversion process, we distinguish between two intuitive types of momentum. One type, called overreaction momentum, posits that the time-0 equity price equals fair value, and the momentum is due to an overreaction by "trend-chasers" to the equity price changes. The result of the overreaction is mispriced equity at time 3.

The other type, called correction momentum, posits that the time-0 equity price differs from fair value, and the momentum is a gradual correction of the mispricing over to fairly-valued equity price at time 3. The correction is due to sharp market professionals exploiting the time-0 mispricing.

In the absence of empirical evidence to the contrary, we assume that both momentum types are equally likely in that it seems equally plausible for equity to be fairly valued at time 0 and mispriced at time 3, as to be mispriced at time 0 and fairly valued at time 3. Although market professionals can identify mispricing, ordinary investors do not know if equity is fairly valued. Nor do the investors know, or care, which type of momentum occurs. Identifying the momentum type only bears on our specification of the reversion process that begins at time 3.

Reversion Process

The overreaction type of momentum is followed by reversion to a trend that is consistent with fairly-valued equity at time 0. For this type of reversion, shown in Exhibit 2, the probability of an up-move (down-move) in the top half of the binomial tree is assumed to be 0.30 (0.70). The bottom half of the tree has the mirror structure for the probability of down- and up-moves. For the time-4 center node, the probability of an up(down)-move is 0.50 (0.50). The assumed up- and down-move probabilities are in boldface.

EXHIBIT 2: REVERSION PROCESS FOR OVERREACTION MOMENTUM

Time:	**3**		**4**		**5**		**6**
							$2.3131
						0.30	
					$2.0114		
				0.30		**0.70**	
			$1.7490				$1.8907
		0.30		**0.70**		**0.30**	
	$1.5209				$1.6441		
		0.70		**0.30**		**0.70**	
			$1.4297				$1.5454
		0.30		**0.70**		**0.30**	
	$1.2432				$1.3438		
		0.70		**0.50**		**0.70**	
			$1.1685				$1.2632
		0.70		**0.50**		**0.70**	
	$1.0161				$1.0984		
		0.30		**0.70**		**0.30**	
			$0.9552				$1.0325
		0.70		**0.30**		**0.70**	
	$0.8306				$0.8979		
		0.30		**0.70**		**0.30**	
			$0.7807				$0.8440
				0.30		**0.70**	
					$0.7339		
						0.30	
							$0.6899

The correction type of momentum is followed by reversion to a trend that is consistent with fairly-valued equity at time 3. This type of reversion, shown in Exhibit 3, emanates from any of the four time-3 equity price outcomes shown in Exhibit 1, with the time-3 equity price rescaled to $1 in Exhibit 3 so that the tree can represent the process emanating from any of those four starting points. At time 3 and in the middle of the tree at time 5, the probability of an up(down)-move is 0.50 (0.50). Otherwise, the probability of an up-move (down-move) in the top half of the binomial tree is assumed to be 0.30 (0.70). The bottom half of the tree has the mirror structure for the probability of down- and up-moves. The assumed up- and down-move probabilities are in boldface.

EXHIBIT 3: REVERSION PROCESS FOR CORRECTION MOMENTUM

Time:	**3**		**4**		**5**		**6**
							$1.5209
						0.30	
					$1.3225		
				0.30		**0.70**	
			$1.15				$1.2432
		0.50		**0.70**		**0.50**	
	$1.00				$1.0810		
		0.50		**0.70**		**0.50**	
			$0.94				$1.0957
				0.30		**0.70**	
					$0.8836		
						0.30	
							$0.8306

Because the two momentum types are assumed to be equally likely, our model's expected reversion process is an equally-weighted blend of the processes in Exhibits 2 and 3. In simulated time series of returns over the model's 6-period horizon, an empiricist would find that single-period (6-month) returns have positive serial correlation on average, whereas three-period (1½-year) returns have negative serial correlation on average. The model's equity price process is thus consistent with the Poterba and Summers (1988) empirical observation that equity returns have positive autocorrelation over short periods and negative autocorrelation over longer periods, and with their conclusion that these patterns suggest transitory deviations in equity price from fair value.[4]

Horizon Outcome Probabilities

At the time-6 horizon, there are seven (7) possible equity price outcomes. Exhibit 4 compares the probabilities of the time-6 outcomes for four equity price processes: (1) overreaction momentum, with the reversion process of Exhibit 2; (2) correction momentum, with the reversion process in Exhibit 3; (3) an equally-weighted blend the two process types, shaded because the blended process is the one used in the study; and (4) a random walk process, where the probability of any single-period equity price move is 0.50.

EXHIBIT 4: TIME-6 EQUITY PRICE PROBABILITY DISTRIBUTIONS

Time-6 Equity Price	Probabilities: Overreaction Momentum	Probabilities: Correction Momentum	Probabilities: Blended Process	Probabilities: Random Walk
2.3131	0.0066	0.0110	0.0088	0.0156
1.8907	0.0532	0.1230	0.0881	0.0938
1.5454	0.1937	0.2390	0.2164	0.2344
1.2632	0.4929	0.2541	0.3735	0.3125
1.0325	0.1937	0.2390	0.2164	0.2344
0.8440	0.0532	0.1230	0.0881	0.0938
0.6899	0.0066	0.0110	0.0088	0.0156

[4] Although Subrahmanyam's (2018) review and synthesis of equity momentum concludes that the cause(s) of momentum are unknown, our model seems reasonable for the purposes of the study in light of Poterba and Summers (1988).

INVESTOR ALLOCATION OPTIONS

The model's investors need to decide how to allocate a lump sum of funds for a three-year horizon. Perhaps the funds are from an inheritance or the sale of a business. The investors choose between DCA and a lump sum buy-and-hold strategy.

The investors cannot specify the details of the expected equity price process, but they intuitively expect future equity prices to follow our model's process. Perhaps the investors developed this intuition through experience and/or acquiring information like "stock prices tend to exhibit momentum in the short run, but revert to fair value in the long run."

Dollar Cost Averaging

The DCA strategy is to accumulate equity during with three (3) equal installments of d, made at times 0, 1, and 2. The choice of three equal DCA installments is based on simplicity, but is otherwise arbitrary. At time 0, d and $1 - d$ are respectively allocated to equity and a 3-year, pure-discount fixed income instrument. At times 1 and 2, d is moved from fixed income (or borrowed if equity is levered) to buy additional equity.

With DCA, the time-6 wealth outcome depends on the four possible paths that equity price can take until time 2. The four possible wealth outcomes at time 2 are each equal to an equity allocation outcome plus the fixed income outcome, where the latter is denoted F_2 and equal to $[(1 - d)1.02 - d]1.02$. For \$1 invested at time 0, the time-2 wealth outcomes and probabilities are shown in Exhibit 5's Panel A.

EXHIBIT 5: DOLLAR COST AVERAGING: WEALTH OUTCOMES & PROBABILITIES

Panel A

Time-2 Wealth Outcomes (W_2) $\quad F_2 = [(1-d)1.02 - d]1.02$	**Probability**
$(d1.15 + d)1.15 + F_2$	0.35
$(d1.15 + d)0.94 + F_2$	0.15
$(d0.94 + d)1.15 + F_2$	0.15
$(d0.94 + d)0.94 + F_2$	0.35

Panel B

Time-6 Wealth Outcomes (W_6) $\quad F_6 = \{[(1-d)1.02 - d]1.02 - d\}1.02^4$

For $W_2 = (d1.15 + d)1.15 + F_2$:

$[(d1.15 + d)1.15 + d]1.15^4$	$+ F_6$	0.0088
$[(d1.15 + d)1.15 + d](1.15^3)0.94$	$+ F_6$	0.0827
$[(d1.15 + d)1.15 + d](1.15^2)0.94^2$	$+ F_6$	0.1458
$[(d1.15 + d)1.15 + d](1.15)0.94^3$	$+ F_6$	0.1049
$[(d1.15 + d)1.15 + d]0.94^4$	$+ F_6$	0.0079

For $W_2 = (d1.15 + d)0.94 + F_2$:

$[(d1.15 + d)0.94 + d]1.15^4$	$+ F_6$	0.0027
$[(d1.15 + d)0.94 + d](1.15^3)0.94$	$+ F_6$	0.0314
$[(d1.15 + d)0.94 + d](1.15^2)0.94^2$	$+ F_6$	0.0819
$[(d1.15 + d)0.94 + d](1.15)0.94^3$	$+ F_6$	0.0314
$[(d1.15 + d)0.94 + d]0.94^4$	$+ F_6$	0.0027

For $W_2 = (d0.94 + d)1.15 + F_2$:

$[(d0.94 + d)1.15 + d]1.15^4$	$+ F_6$	0.0027
$[(d0.94 + d)1.15 + d](1.15^3)0.94$	$+ F_6$	0.0314
$[(d0.94 + d)1.15 + d](1.15^2)0.94^2$	$+ F_6$	0.0819
$[(d0.94 + d)1.15 + d](1.15)0.94^3$	$+ F_6$	0.0314
$[(d0.94 + d)1.15 + d]0.94^4$	$+ F_6$	0.0027

For $W_2 = (d0.94 + d)0.94 + F_2$:

$[(d0.94 + d)0.94 + d]1.15^4$	$+ F_6$	0.0079
$[(d0.94 + d)0.94 + d](1.15^3)0.94$	$+ F_6$	0.1049
$[(d0.94 + d)0.94 + d](1.15^2)0.94^2$	$+ F_6$	0.1458
$[(d0.94 + d)0.94 + d](1.15)0.94^3$	$+ F_6$	0.0827
$[(d0.94 + d)0.94 + d]0.94^4$	$+ F_6$	0.0088

For each time-2 wealth outcome, there are five different time-6 wealth outcomes, with each equal to a time-6 equity allocation outcome plus the fixed income outcome, where the latter is denoted F_6 and equal to $\{[(1-d)1.02-d]1.02-d\}1.02^4$. The twenty (20) possible time-6 DCA wealth outcomes and their probabilities are shown in Exhibit 5's Panel B.[5]

Lump Sum Buy-and-Hold

The buy-and-hold strategy is to plunge the funds at time 0 into an equity and fixed income allocation for the 3-year horizon. The time-0 percentage equity allocation is denoted w; the rest of the time-0 investable wealth is allocated to a 3-year, pure-discount fixed income instrument, which is a negative allocation for levered equity holders. There are seven (7) possible time-6 wealth outcomes, one for each of the time-6 equity price outcomes in Exhibit 4. For \$1 of time-0 investable wealth, each time-6 wealth outcome for the buy-and-hold strategy is equal to w times the time-6 equity price outcome plus $(1-w)$ times 1.02^6 in fixed income.

Investor Utility and Optimal Allocations

The model assumes that investors have a traditional power utility function of the form $U = 1 - W^{1-b}$, where W denotes horizon wealth, b denotes an investor's degree of constant relative risk aversion (CRRA), and initial wealth is normalized to \$1. If $b = 1$, the utility function is the log utility function, $U = lnW$. Guo and Whitelaw (2006) indicate that power utility describes empirical data fairly well and that the average investor's CRRA is above 1.

For a given b, an investor's optimal allocation plan maximizes the expected utility of possible time-6 wealth outcomes. We calculate the optimal time-0 equity allocation for the buy-

[5] To clarify, the 20 outcomes in Panel B reflect all 7 of the possible time-6 equity prices listed in Exhibit 4. Although there are only 5 possible terminal equity prices that can emanate from the time-2 equity price for each of the four paths, each set of 5 terminal outcomes incorporates a different subset of the 7 possible terminal equity prices.

and-hold strategy, denoted w^*, by trial and error. Although real-world investors are not likely to perform this calculation, we assume that investors know their optimal buy-and-hold allocation intuitively, based on experience and personal tolerance for risk.

Two options are considered for DCA. The first is for the three d installments to sum to w^*. This option is called naïve DCA, and denoted as d^*. The other option is to find the optimal 3-installment d, denoted d^O, which maximizes the expected utility of time-6 wealth. We calculate d^O by trial and error. The first option is a natural default level for d for an investor who has never implemented DCA. The second option represents a d level that an investor might have learned from implementing DCA in the past.

To get an economic interpretation of the difference between the strategies' expected utilities, we compare the certain-wealth equivalent of the expected utility, following Larson and Munk (2012).

Optimal Reallocation vs Optimal Buy-and-Hold

In the Samuelson (1991) model, an investor who optimally reallocates each period expects a higher utility of horizon wealth than by either implementing DCA or plunging a lump sum into the optimal buy-and-hold strategy. However, to get the higher expected utility, the investor must develop the optimal reallocation plan using the process probabilities. For a random walk, the plan is easy because each period's allocation is the same and equal to a single-period allocation. More generally, however, the traditional approach to an optimal reallocation plan involves dynamic programming, first finding the optimal single-period allocation going forward from each time-5 outcome. Then one works backward to find the optimal 2-period allocation for each time-4 outcome, and so forth until finally finding the optimal 6-period initial allocation at time 0.

Although the optimal reallocation plan yields a higher expected utility than either DCA or the optimal buy-and-hold strategy, it is unlikely that a typical real-world investor would attempt to determine the complex optimal reallocation plan for our model's equity price process.[6] It is just as unlikely that such an investor would implement the plan, which would call for buying (selling) equity in upward (downward) trends during momentum and doing the opposite during reversion. Moreover, the reallocations can involve large swings into levered equity positions and short equity positions, which is difficult to imagine for a typical real-world investor. Therefore, the study asserts that a typical real-world investor's alternative to DCA is more realistically the optimal buy-and-hold strategy than the optimal reallocation strategy.

EXAMPLE

This section provides an example of an investor with b = 4. For \$1 of time-0 investable wealth and the blended equity process, we first use information in Exhibit 4 to find the w^* that maximizes the expected utility of time-6 wealth for the 3-year buy-and-hold strategy:

$$U(W_6) = 0.0088\{1 - [w^*2.3131 + (1 - w^*)1.02^6]^{-3}\} + 0.0881\{1 - [w^*1.8907 + (1 - w^*)1.02^6]^{-3}\} + 0.2164\{1 - [w^*1.5454 + (1 - w^*)1.02^6]^{-3}\} + 0.3735\{1 - [w^*1.2632 + (1 - w^*)1.02^6]^{-3}\} + 0.2164\{1 - [w^*1.0325 + (1 - w^*)1.02^6]^{-3}\} + 0.0881\{1 - [w^*0.8440 + (1 - w^*)1.02^6]^{-3}\} + 0.0088\{1 - [w^*0.6899 + (1 - w^*)1.02^6]^{-3}\}.$$

The expected utility is maximized with w^* = 0.68 (found by trial and error). Therefore, an investor with b = 4 has an optimal time-0 3-year buy-and hold allocation of 68% in equity and

[6] Breeden (2004) shows an alternative analytical approach that is simpler than dynamic programming but is still too sophisticated to be applied by typical real-world investors.

32% in fixed income. The maximum expected utility is 0.3922, which has a certain-wealth equivalent of $1.1806.[7]

Assume next that the investor implements naïve DCA with $d^* = w^*/3$ = 0.68/3 = 0.227. For an example of a time-6 wealth outcome, the highest is when the realized time-6 equity price is $2.3131, and is equal to $\{[(0.227(1.15) + 0.227)1.15] + 0.227\}1.15^4 + \{[((1 - 0.227)1.02 - 0.227)1.02] - 0.227\}1.02^4$ = $1.7519. For b = 4, the expected utility of the 20 time-6 wealth outcomes is 0.3922, which is calculated using the information in Exhibit 5. The certain-wealth equivalent is $1.1806. Thus, naïve DCA yields an expected utility of time-6 wealth equal to that for the optimal lump sum strategy.

Next assume that the investor instead implements optimal DCA, where the installment level, d^O, maximizes the expected utility of time-6 wealth. As found by trial and error, the investor's d^O = 0.293. For an example of a time-6 wealth outcome, the highest is when the realized time-6 equity price is $2.3131, and is equal to $\{[(0.293(1.15) + 0.293)1.15] + 0.293\}1.15^4 + \{[(1 - 0.293)1.02 - 0.293]1.02 - 0.293\}1.02^4 = \$1.9360$. The expected utility for optimal DCA is 0.3968, which has a certain-wealth equivalent of $1.1835. Therefore, optimal DCA is better than the optimal lump sum strategy by $1.1835 – 1.1806 = $0.0029, or 29 basis points.

RESULTS

This section summarizes a comparison of DCA and the lump sum strategy in our fundamental dynamic allocation model. The main results, for a range of risk aversion levels from

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[7] The investor's time-0 equity allocation of 68% is higher than it would be for a random walk, which we show later is 60%. This difference illustrates Samuelson's (1991) "horizon effect": the investor's w^* is higher for the momentum/reversal process because the distribution of horizon wealth outcomes has less dispersion than for a random walk. Note that with optimal reallocation, the horizon effect for $b > 1$ would also have a component related to hedging the uncertainty in future periods, in the sense of Merton (1973).

$b = 1$ to $b = 12$, are presented in the first subsection. Following the second subsection's discussion of the main results, the third subsection provides comparative results for a random walk process. To obtain additional insights, the fourth subsection compares the results for an investor with $b = 4$ for a range of different strengths of the assumed momentum and reversion processes, where process strength is measured by the up- and down-move probabilities, holding the equity price movements the same. The final subsection looks at the impact of equity volatility level by changing the single-period equity price relatives, while holding process strength (probabilities) the same.

Main Results

Exhibit 6, which shows the main results, has sections for each of three allocation options: (1) the optimal lump sum strategy, (2) naïve DCA, and (3) optimal DCA. For each allocation option and a range of investor risk aversion levels, Exhibit 6 shows the time-0 equity allocation and the certain-wealth equivalent of the expected time-6 utility per \$1 of time-0 investment. Each DCA option also shows the difference between the certain-wealth equivalents for DCA and the optimal lump sum strategy.

Exhibit 6's optimal lump sum section first shows the optimal time-0 3-year buy-and-hold equity allocation, w^*. For investors with $b = 2.5$ and lower, $w^* > 1$ in our model. These more risk tolerant investors hold levered equity portfolios. The more risk averse investors with $b > 2.5$ hold traditional long-run portfolios with positive allocations in both equity and fixed income.[8]

[8] In static capital market theory, the risk-free asset is in zero net supply, with less risk-averse investors borrowing from more risk-averse investors. An investor who holds 100% in equities is the "representative investor". Our model's assumed equity process and constant risk-free rate of 2% per half-year period would imply that the representative buy-and-hold investor's risk aversion is close to $b = 2.5$.

EXHIBIT 6: RESULTS FOR BLENDED EQUITY PRICE PROCESS

	Optimal Lump Sum		Naïve DCA			Optimal DCA		
Risk Aversion b	**Buy-and-Hold w^***	**Certain-Wealth Equivalent**	**Naïve DCA d^***	**Certain-Wealth Equivalent**	**DCA – Lump (bps)**	**Optimal DCA d^0**	**Certain-Wealth Equivalent**	**DCA – Lump (bps)**
1	2.29	1.3464	0.763	1.3355	–109	0.923	1.3490	27
1.50	1.73	1.2762	0.577	1.2729	–33	0.730	1.2816	54
2	1.34	1.2380	0.447	1.2367	–13	0.583	1.2430	50
2.50	1.09	1.2149	0.363	1.2145	–4	0.463	1.2192	43
3	0.91	1.1995	0.303	1.1993	–2	0.390	1.2033	38
3.50	0.78	1.1886	0.260	1.1886	0	0.343	1.1920	34
4	0.68	1.1806	0.227	1.1806	0	0.293	1.1835	29
5	0.54	1.1693	0.189	1.1693	0	0.233	1.1717	24
6	0.45	1.1619	0.150	1.1619	0	0.193	1.1639	20
8	0.33	1.1528	0.110	1.1528	0	0.147	1.1543	15
10	0.27	1.1474	0.090	1.1474	0	0.117	1.1491	12
12	0.22	1.1438	0.073	1.1438	0	0.097	1.1448	10

Exhibit 6 shows that naïve DCA is inferior to the optimal lump sum strategy for investors with $b < 3$, but the choice does not matter for higher levels of risk aversion. For all investors, the optimal DCA installment is higher than the naïve DCA one, and optimal DCA is better than the optimal lump sum strategy. For optimal DCA, the largest difference in certain-wealth equivalent is 54 basis points, for $b = 1.5$. For $b > 1.5$, the basis point advantage for optimal DCA drops as the equity allocation percentage drops. For the highly-levered log utility investor, with $b = 1$, optimal DCA is superior by 27 basis points.[9]

[9] Note that optimal DCA may result in levered equity even if $w^* < 1$.

Discussion of Main Results

To better appreciate the Exhibit 6 results, it is instructive to first compare the impact of each momentum process type for an investor with b = 4. If the momentum process were exclusively the overreaction type, w^* = 0.86 and d^O = 0.357, and naïve (optimal) DCA would be inferior by 17 basis points (superior by 11 basis points) to the optimal lump sum. If the momentum process were exclusively the correction type, w^* = 0.55 and d^O = 0.247, and naïve (optimal) DCA would be superior by 2 (31) basis points to the optimal lump sum. Therefore, the correction type process contributes somewhat more than the overreaction type process to DCA's positive performance in the Exbibit 6 results.

Because correction type momentum corresponds to mispriced equity at time 0, the positive findings for optimal DCA support the intuitive justification often given for DCA, that it avoids plunging a lump sum into equity at a price that might be temporarily too high. If empirical research were to find that correction type momentum is more prevalent than overreaction type momentum, or if an investor believes that equity is more likely to be mispriced than fairly valued at time 0, the case for DCA would be strengthened.

Next, we compare the relative impact of momentum versus reversion. A pure momentum process for all six periods is created by changing Exhibit 2's probabilities from 0.30 (0.70) to 0.70 (0.30) and combining the resulting binomial tree with Exhibit 1. A pure reversion process is created by changing Exhibit 1's 0.30 (0.70) probabilities to 0.70 (0.30) and combining the resulting tree with Exhibit 2. Therefore, the comparison converts the overreaction momentum process, where w^* = 0.86, to a pure momentum or pure reversion process.

Consider an investor with b = 4 who either allocates 0.86 to equity in a buy-and-hold strategy or implements naïve DCA with three equal installments of 0.287. For a pure momentum process, naïve DCA is better by 289 basis points. For a pure reversion process, the optimal lump sum strategy is better by 120 basis points. This example illustrates that momentum rather than reversion is driving DCA's positive performance. The reason is that DCA's reduction of time-6 wealth outcome dispersion is more valuable for a more disperse distribution of equity prices.

However, note that the preceding paragraph assumes that the investor's lump sum allocation (0.86) is optimal for the overreaction type of equity process. Fixing that allocation served to isolate the impact of momentum or reversion on the performance of DCA. But for the optimal lump sum allocation under each pure process, DCA is inferior to the optimal lump sum strategy.[10] Hence, for DCA to perform better than the optimal lump sum strategy, the equity price process needs have both momentum and reversion, as in our model.

Random Walk

If equity prices follow a random walk, the probability of a single-period up- or down-move is 0.50. Exhibit 7 shows the results of comparing the three investment options for a random walk process. As indicated by the negative numbers in the third column of both DCA sections, Exhibit 7 shows that the optimal lump sum strategy is better than either DCA option for every level of risk aversion.[11]

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[10] For b = 4 with the pure momentum process, for example, w^* = 0.32, d^O = 0.127, and the optimal lump sum strategy is better than naïve (optimal) DCA by 32 (26) basis points in certain-wealth equivalent. For b = 4 with the pure reversal process, w^* = 1.11, d^O = 0.403 and the optimal lump sum strategy is better than naïve (optimal) DCA by 110 (106) basis points in certain-wealth equivalent. The results of this comparison are qualitatively similar using the correction type process or the blended process.

[11] Of course, we already know that the three allocation options are suboptimal to the constant allocation with period-by-period rebalancing. For reasonable equity volatility levels, however, it is well known (e.g., Barbaris, 2000) that optimal buy-and-hold allocations are close to optimal for investors with positive amounts of equity and fixed income. So, Exhibit 7's results are not surprising. However, the results help put into perspective the relative cost of using DCA if equity were to follow a random walk.

The difference between the certain-wealth equivalents of the optimal lump sum strategy and naïve (optimal) DCA is 215 (174) basis points for the log investor (b = 1). As b gets higher, the difference drops. For an investor with b = 4, with a buy-and-hold allocation of w^* = 0.60 (60% equity), the difference between the certain-wealth equivalents of the optimal lump sum strategy and naïve (optimal) DCA is 45 (40) basis points. Investors are better off with the optimal lump sum plunge when the equity price process is a random walk because DCA's delayed equity allocation is less valuable for a random walk than for our model's process.

EXHIBIT 7: RESULTS FOR A RANDOM WALK PROCESS

	Optimal Lump Sum		Naïve DCA			Optimal DCA		
Risk Aversion b	**Buy-and-Hold w^***	**Certain-Wealth Equivalent**	**Naïve DCA d^***	**Certain-Wealth Equivalent**	**DCA – Lump (bps)**	**Optimal DCA d^O**	**Certain-Wealth Equivalent**	**DCA – Lump (bps)**
1	2.14	1.3304	0.713	1.3089	–215	0.810	1.3130	–174
1.50	1.57	1.2636	0.523	1.2501	–135	0.587	1.2517	–119
2	1.20	1.2283	0.400	1.2185	–98	0.447	1.2196	–87
2.50	0.95	1.2070	0.317	1.1992	–78	0.360	1.2000	–70
3	0.80	1.1930	0.267	1.1867	–63	0.300	1.1875	–55
3.50	0.69	1.1831	0.230	1.1779	–52	0.257	1.1784	–47
4	0.60	1.1757	0.200	1.1712	–45	0.223	1.1717	–40
5	0.48	1.1655	0.160	1.1619	–35	0.177	1.1623	–32
6	0.40	1.1587	0.133	1.1558	–29	0.147	1.1561	–26
8	0.30	1.1504	0.100	1.1483	–21	0.110	1.1485	–19
10	0.24	1.1455	0.080	1.1438	–17	0.087	1.1439	–16
12	0.20	1.1422	0.067	1.1408	–14	0.073	1.1409	–13

The results for the random walk process help us to see that the momentum and/or reversion in our model's process must have sufficient strength for DCA to be better than the optimal lump sum strategy. If momentum (reversion) has a low (high) enough up-move probability in the top half of the binomial tree, the process approaches a random walk. This issue bears the further investigation summarized in the next subsection.

Equity Process Strength and Dollar Cost Averaging

This subsection presents results for various equity process strengths, measured by the binomial probabilities and holding the equity price movements the same. For an investor with b = 4, Exhibit 8 shows results for different strengths for momentum and reversion. The rows show the momentum strength measured by up-move/down-move probabilities in the top half of the binomial tree in Exhibit 1. The strength ranges from very strong momentum, 0.90/0.10, to moderate momentum, 0.60/0.40; the last row is for a random walk, 0.50/0.50. The columns show the reversion strength in Exhibits 2 and 3, ranging from very strong reversion, 0.10/0.90, to moderate reversion, 0.40/0.60. The last column is for a random walk, 0.50/0.50.

Each Exhibit 8 cell shows four items for an investor with b = 4. The top two items are the optimal lump sum equity allocation, w^*, and the optimal DCA installment, d^O, separated by a slash. The bottom two items are the differences between certain-wealth equivalents of DCA and the optimal lump sum strategy, in basis points; before the slash is for naïve DCA, and after the slash is for optimal DCA. For example, for medium momentum, 0.70/0.30, combined with medium reversion, 0.30/0.70, the optimal buy-and-hold equity allocation is w^* = 0.68 and the optimal DCA installment is d^O = 0.293. Naïve (optimal) DCA's certain-wealth equivalent the same as the same as (29 basis points higher than) that for the optimal lump sum strategy, as calculated in the Example section and shown shaded in Exhibit 6. This information is in Exhibit 8's middle cell, with bold numbers and borders. Exhibit 7's random walk information for a b = 4 investor is in Exhibit 8's cell for the last column and bottom row, which also has bold borders.

Exhibit 8's cells are dark-shaded if both naive and optimal DCA are better than the optimal lump sum strategy and are light-shaded if only optimal DCA is better.

The results in Exhibit 8 show that for a given reversion strength, DCA's performance improves in the momentum strength. Moreover, for a given momentum strength, DCA's performance improves in the reversion strength. For strong or very strong momentum, optimal DCA is better even if the reversion strength approaches a random walk.

EXHIBIT 8: RESULTS OF PROCESS STRENGTH COMBINATIONS FOR *b* = 4

REVERSION →	**Very Strong Reversion 0.10/0.90**	**Strong Reversion 0.20/0.80**	**Medium Reversion 0.30/0.70**	**Moderate Reversion 0.40/0.60**	**Random Walk 0.50/0.50**
MOMENTUM ↓	w^*/d^0 **DCA – Lump**	w^*/d^0 **DCA – Lump**	w^*/d^0 **DCA – Lump**	w^*/d^0 **DCA – Lump**	w^*/d^0 **DCA – Lump**
Very Strong Momentum 0.90/0.10	0.71/0.340 28/88	0.61/0.300 21/78	0.52/0.250 8/53	0.45/0.207 –3/28	0.39/0.173 –14/6
Strong Momentum 0.80/0.20	0.81/0.377 22/79	0.70/0.323 19/68	0.59/0.270 6/45	0.50/0.223 –7/22	0.43/0.183 –16/2
Medium Momentum 0.70/0.30	0.93/0.417 10/60	0.80/0.353 8/50	**0.68/0.293 0/29**	0.57/0.240 –11/10	0.48/0.200 –21/–6
Moderate Momentum 0.60/0.40	1.06/0.453 –15/29	0.92/0.383 –11/19	0.77/0.317 –17/4	0.64/0.260 –25/–10	0.54/0.210 –30/–20
Random Walk 0.50/0.50	1.22/0.493 –52/–20	1.04/0.410 –47/–26	0.87/0.337 –45/–32	0.72/0.277 –45/–37	0.60/0.223 –45/–40

Equity Volatility and Dollar Cost Averaging

For a given equity process strength, the impact of equity volatility may be gauged by varying the up- and down-move equity price change assumption. For the equity up(down)-move of 1.15(0.94), an investor with b = 4 was shown to be 29 basis points better off with optimal DCA, given an equity price process with medium momentum and medium reversion.

For an investor with b = 4, Exhibit 9 shows some outcomes for various equity price volatility assumptions, holding all else the same. The first and second columns in Exhibit 9 show the assumed single-period up- and down-move equity price relatives. The third column shows the corresponding annualized equity return volatility for single-period up- and down-move probabilities of 0.50 and 0.50.

For an equity price process that combines medium momentum and medium reversion, the first column in Exhibit 9's middle panel shows the optimal buy-and-hold allocation, w^*, and the optimal DCA installment, d^O, separated by a slash. The next column shows the certain-wealth equivalent difference between optimal DCA and the optimal lump sum strategy. In Exhibit 9, DCA's economic benefit over the lump sum strategy is higher for lower equity volatility, all else the same. This finding seems counter-intuitive but is because holding b the same implies a higher optimal buy-and-hold equity allocation when volatility is lower.

For a random walk process, Exhibit 9's right panel shows that the lump sum strategy's advantage over DCA drops with higher volatility. At higher equity volatility, the economic cost is lower for an investor who uses optimal DCA instead of the lump sum strategy.

EXHIBIT 9: IMPACT OF EQUITY PRICE VOLATILITY

VOLATILITY LEVEL			MEDIUM MOMENTUM/ MEDIUM REVERSAL		RANDOM WALK	
Equity Up-Move	Equity Down-Move	Annualized Volatility	Optimal w^*/d^0	DCA – Lump (bps)	Optimal w^*/d^0	DCA – Lump (bps)
1.13	0.96	0.120	1.05/0.457	56	0.93/0.347	–65
1.14	0.95	0.134	0.83/0.363	41	0.74/0.277	–51
1.15	0.94	0.144	0.68/0.293	29	0.60/0.223	–40
1.16	0.93	0.163	0.56/0.240	21	0.50/0.187	–33
1.17	0.92	0.177	0.46/0.203	14	0.42/0.157	–27
1.18	0.91	0.191	0.39/0.170	9	0.35/0.133	–23
1.19	0.90	0.205	0.34/0.147	5	0.31/0.113	–19
1.20	0.89	0.219	0.29/0.123	1	0.26/0.100	–16

CONCLUSION

This study compares dollar cost averaging (DCA) with an optimal buy-and-hold strategy in a fundamental dynamic allocation model where the equity price process has momentum and reversion. The comparison shows that with sufficient strength in the momentum and reversion, DCA may be better than the lump sum strategy for investors with traditional utility functions and a wide range of constant relative risk aversion levels. Whether real-world equity price processes have the sufficient momentum and reversion strength to justify DCA is an empirical question that we leave to researchers with skills in empirical methods.

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