The St. Petersburg Paradox and Capital Asset Pricing

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ABSTRACT

Durand (1957) shows that the classical St. Petersburg paradox can apply to the valuation of a firm whose dividends grow at a constant rate forever. To capture a more realistic pattern of dividends, we model the dividend growth rate as a mean reverting process, and then use the CAPM to derive the risk-adjusted present value. The model generates an equivalent St. Petersburg game. The long-run growth rate of the payoffs (dividends) is dominant in driving the value of the game (firm), and the condition under which the value is finite is less restrictive than that of the standard game.

Keywords: Asset pricing; Dividends; St. Petersburg paradox

JEL Classification**:** G0, G1

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1. Introduction

The St. Petersburg paradox is one of the most well-known and interesting problems in the history of financial economics. The paradox describes a situation where a simple game of chance offers an infinite expected payoff, and yet any reasonable investor will pay no more than a few dollars to participate in the game. Since the paradox was presented by Daniel Bernoulli in 1738, it has attracted a great deal of interest, mainly by theorists who provide solutions and derive its implications.

One of the applications of the paradox is in the area of financial asset pricing. Durand (1957) shows that the St. Petersburg game can be transformed to describe a conventional stock pricing model for growth firms. The analogy is based on the assumption that the firm's future dividends (as the game's future payoffs) grow at a constant rate g. Economic intuition and the historical evidence suggest, however, that the very high growth rates experienced by many young firms (e.g., firms in the high-tech industry) are expected to decline over time. Hence, the short-run growth rate is typically much greater than the expected long-run rate. In this study we model the dividend growth rate as a mean reverting process; we then find the risk-adjusted growth rate under the equilibrium setting of the classical Capital Asset Pricing Model (CAPM), and derive its equivalent modified St. Petersburg game.[1]

Our paper has several interesting results. First, we assume an autoregressive process for the dividend growth rate, and then use the CAPM to derive a closed form solution for the price of a growth stock. Second, by distinguishing between the short-run and long-run growth rates, the model shows that the latter is the dominant factor in driving the properties of the St. Petersburg game (or the firm), including its value. Third, and last, we derive the condition under which the expected payoff of the game (or equivalently, the value of a growth firm) is finite; as expected,

[1] Assuming a stochastic dividend growth is rather common in asset pricing studies; see, for example, Bansal and Yaron (2004), and Bhamra and Strebulaev (2010).

this condition is much less restrictive than the one required under constant growth. This result provides an indirect solution to the game; we show that if the payoffs' growth rate is mean reverting, then the present value is finite, and the fee players are willing to pay is finite.

2. The paradox and its common solutions

The St. Petersburg paradox is based on the following simple game. A fair coin is tossed repeatedly until the first time it falls on 'head'. The player's payoff is 2^n dollars ('ducats' in the original Bernoulli's paper), where n is the number of tosses. Since n is a geometric random variable with p=0.5, the expected payoff of the game is:

$$E = \frac{1}{2}\times 2 + \frac{1}{4}\times 4 + \frac{1}{8}\times 8 + ... = 1 + 1 + 1 + ... = \infty \tag{1}$$

Yet, while this game offers an expected payoff of infinite dollars, a typical player will pay no more than a few dollars to participate in the game, reasoning that there is a very small probability to earn a significant amount of money. For example, the chance to earn at least 32 dollars is $2^{-5} = 0.03125$, and at least 128 dollars is $2^{-7} = 0.00713$; hence, paying a game fee of even 1,000 dollars seems unreasonable.

A number of solutions proposed to resolve the paradox rely on the concept of utility.[2] The basic idea is that the relative satisfaction from an additional dollar decreases with the total amount of money received. Thus, the game fee should be based on the expected utility from the dollars earned, rather than the expected amount of dollars. Bernoulli himself suggested the log utility function; in this case, the expected value of the game is:

$$E(U) = \frac{1}{2}\times \ln(2) + \frac{1}{4}\times \ln(4) + \frac{1}{8}\times \ln(8) + ... = \sum_{n=1}^{\infty} \frac{\ln(2^n)}{2^n} = 2\ln(2) < \infty \tag{2}$$

[2] See Senetti (1976) on solving the paradox using expected utility in light of modern portfolio theory, and Sz´ekely and Richards (2004) for other suggested solutions to the paradox.

It turns out, however, that the utility-based solutions are incomplete: the game can be converted to a convex stream of payoffs, and this change reverses the benefit of a concave utility function. For example, if instead of 2^n, the game's payoff is e^{2^n}, then for the log utility just considered, the expected value again tends toward infinity. In general, for every unbounded utility function the payoffs can be changed such that the expected utility will be infinite. This generalization of the game often goes by the name "super St. Petersburg paradox."

Proposed solutions to the super game rely on the concept of risk aversion (Friedman and Savage (1948), and Pratt (1964)); that is, holding everything else constant, a typical player prefers less risk, and therefore is willing to pay a lower fee for high-risk games. Weirich (1984) shows that, while the game may offer an expected infinite sum of money, it involves also an infinite amount of risk. This is because the dispersion of the possible payoffs results in an infinite standard deviation. Thus, the fair game fee resembles the difference between an infinite expected payoff and an infinite measure of risk. While the answer may actually be finite, it does not explain the specific amounts typical players are willing to pay, which range between 2 to 25 dollars. (The risk-aversion assumption is central for our analysis of the paradox in the CAPM environment in Section 4). In sum, it appears that none of the solutions offered so far provides a complete answer to all aspects of the problem.

3. Application of the paradox to growth stocks

One of the applications of the St. Petersburg paradox is in the area of asset pricing. Durand (1957) shows that with some modifications, the paradox can describe a conventional stock pricing model. A growth firm expects to generate an increasing stream of future earnings, and thus to pay an increasing stream of dividends. The value of such a firm is given by the present value of all future dividends:

$$P_0 = \frac{D_1}{(1+r)} + \frac{D_2}{(1+r)^2} + ... = \sum_{t=1}^{\infty} \frac{D_t}{(1+r)^t} \tag{3}$$

where D_t is the per-share dividend of year *t* and *r* is the discount rate. The constant growth model, also known as the Gordon (1962) model, assumes that future dividends grow at a constant rate (*g*); i.e., the dividend stream is $D_1, D_1(1+g), D_1(1+g)^2, ...$ for the years 1, 2, 3,… In that case, the present value of this growing perpetual future dividend stream equals:

$$P_0 = \sum_{t=1}^{\infty} \frac{D_1(1+g)^{t-1}}{(1+r)^t} = \begin{cases} \frac{D_1}{r-g}, & if \;\; g < r \\ \infty, & if \;\; g \geq r \end{cases} \tag{4}$$

Thus, the firm value is finite only if the growth rate is lower than the discount rate.

Durand derives the St. Petersburg analogue of the constant growth model using the following analysis. Consider the St. Petersburg game, where instead of a fair coin, the probability that a 'head' appears is $r/(1+r)$, where $r > 0$. Assume further that instead of earning a single payment when the game ends (i.e., when 'head' appears in the first time), the player earns a specific amount of dollars as long as the game continues. Specifically, the player will earn D_1 if the first toss is 'tail', $D_1(1+g)$ if the second toss is 'tail', $D_1(1+g)^2$ if the third toss is 'tail', and so on. That is, if the game lasts for *n* tosses, instead of earning 2^n, the player will earn: $\sum_{j=1}^{n-1} D_1(1+g)^{j-1} = \frac{D_1[(1+g)^{n-1}-1]}{g}$. Therefore, the expected payoff of the game is:

$$E = \sum_{n=1}^{\infty} \frac{r}{(1+r)^n} \frac{D_1[(1+g)^{n-1}-1]}{g} = \sum_{n=1}^{\infty} \frac{D_1(1+g)^{n-1}}{(1+r)^n} = \begin{cases} \frac{D_1}{(r-g)}, & if \;\; g < r \\ \infty, & if \;\; g \geq r \end{cases} \tag{5}$$

which is identical to the value of a constant dividend growth firm (as appears in Equation 4).

Note that the analogy is based on the translation of the discount rate *r* to the coin probability $r/(1+r)$. That is, while in the original St. Petersburg game any future payment will be paid with some probability, in the modified game, which is equivalent to the dividend stream generated by growth firms, any future payment will be paid for sure, but will be evaluated with a discount factor. This analogy helps explaining the condition under which the expected payoff of the game is infinite. The value of the stock is infinite only when the dividend grows at an equal or higher rate than the discount rate; and in the same way, the expected payoff of the game is infinite only when the payoffs are increasing at an equal or higher rate than the rate at which the correspondent probabilities are decreasing.

We believe the paradox arises for two reasons. First, the assumption that the dividend stream will grow at a constant rate permanently is unrealistic. For example, economic intuition, as well as the historical evidence, suggests that high growth tech firms (such as IBM, Microsoft and now Google) may grow very rapidly in the short run. But nothing attracts competition like market success; therefore, in the long run new market entrants (e.g., Google) will force the earnings growth rate to slow down (almost surely) to a level consistent with the growth rate of the overall economy. The second reason is that the degree of risk implicit in the dividend stream may cause investors to change the risk-adjusted discount rate, i.e., the probability of actually receiving the expected dividends.

In the next section we formalize these ideas within the context of the Capital Asset Pricing Model (CAPM) of Sharpe (1964) and Lintner (1965). We model the dividend growth rate as a mean reverting process so that the current rate can be very large but the long run rate is expected to be much lower. We then use the CAPM to derive the appropriate risk-adjusted present value. We find that the equity price can be finite without the unreasonable condition required by the constant growth model.

4. Stochastic dividend valuation in the CAPM world

Let D_t be the time-*t* value of dividends or earnings (for simplicity we will use these two terms interchangeably), and assume these values grow at rate g_{t+1}: $D_{t+1} = (e^{g_{t+1}})D_t$. We use a first order autoregressive process (AR(1)) to model mean reversion:

$$g_{t+1} = (1-\phi)\overline{g} + \phi g_t + \varepsilon_{t+1} \tag{6}$$

Where $\overline{g}$ is the long run (unconditional) mean growth rate and ϕ is the autoregressive coefficient. We make the usual assumptions to insure the process is stationary and the growth rate is mean reverting. The innovation terms ε_{t+1} are normally distributed random variables with mean zero, variance σ_ε^2, no serial correlation, and constant covariance with the market portfolio. The major implication of the AR model is that while the current growth rate can be abnormally large, in the long run earnings growth should slow down to a lower rate. Intuitively, we expect $\overline{g}$ to be close to the growth rate for the overall economy because of competitive pressures brought about by new startup companies.

To obtain the risk-adjusted present value of each future dividend we use the CAPM of Sharpe (1964) and Lintner (1965):

$$ER = R_f + [ER_m - R_f]\beta_{ROR} \tag{7}$$

where *ER* is the single period return on the asset, R_f is the risk-free rate of interest, ER_m is the expected market portfolio return, and market risk is measured by the rate of return beta (β_{ROR}).

Then, the CAPM price for an asset with a stochastic dividend stream $\{D_t\}_{t=1}^{\infty}$ is given by the sum of expected dividend values adjusted, for market risk, and discounted to the present at the riskless rate of interest (Equation (8) is derived in the Appendix):

$$P_0 = \sum_{t=1}^{\infty} \frac{(E_0\, D_t)\prod_{j=1}^{t}\left[1-(ER_m - R_f)z_j\beta_g\right]}{\left(1+R_f\right)^t}$$

$$= \sum_{t=1}^{\infty} \frac{D_0\, e^{(1-\phi)\bar{g}\sum_{j=1}^{t} z_j + z_t\phi g_0 + (\sigma_\varepsilon^2/2)\sum_{j=1}^{t} z_j^2 + \sum_{j=1}^{t} \ell n\left[1-(ER_m - R_f)z_j\beta_g\right]}}{\left(1+R_f\right)^t} \qquad (8)$$

The deterministic variable z_j captures the effect of mean reversion on cash flows and the market risk. It may be computed recursively as: $z_j = \phi z_{j-1} + 1$ for j=1, 2, ... and starting value $z_0 = 0$. The growth rate beta, β_g, is defined as the covariance between the growth rate innovation (ε_{t+1}) and the market return, divided by the variance of the market return.

Equation (8) shows that the current price depends on the current growth g_0, and the long run rate $\bar{g}$; however, the impact of the latter is much stronger because it is multiplied by the sum of z_j, which is always positive. Intuitively, stronger mean reversion implies faster reversal to the long run mean; therefore, a currently high growth rate has only a transitory impact on the equity price. Today's price depends also on the appropriate risk adjustment. Since the market risk, represented by $z_j\beta_g$ increases with j -- up to $\beta_g/(1-\phi)$ as j approaches infinity, the adjustment for risk becomes increasingly large for distant expected dividends, and this helps obtain a finite present value.

Two special cases of the general model are worth mentioning. The first is Gordon's deterministic growth model which is obtained by setting $\phi = 0$ and $\sigma_\varepsilon^2 = 0$. In this case, dividends are expected to grow in a deterministic fashion at a constant rate $\bar{g}$. The beta factor (β_g) equals zero and the discount rate r equals the riskless rate R_f.

The second, originally developed by Rubenstein (1976) within the context of a single factor arbitrage-free model, allows stochastic growth but no serial correlation: $g_{t+1} = \bar{g} + \varepsilon_{t+1}$. In this case, (log) earnings or dividends follow a random walk with drift, and the stock price has a closed-form solution similar to Durand's formula: $P_0 = \frac{D_0 e^{g^*}}{r - e^{g^*}}$, where the discount rate r is given by $\frac{(1+R_f)}{1-(ER_m - R_f)\beta_g}$, and the adjusted growth rate is $g^* = \bar{g} + \sigma_\varepsilon^2 / 2$. The major drawback of these two models is that unlike our autoregressive model, they do not allow a distinction between current growth – which can be abnormally high, and long-run growth.

We can now derive the condition under which the asset price is finite. Observe that for large j, z_j converges to a constant value of $1/(1-\phi)$; therefore, the expected future dividend will evolve along the path $A_T = e^{\left(\bar{g} + \frac{1}{2}\frac{\sigma_\varepsilon^2}{(1-\phi)^2}\right)T}$. Also, for dividends far into the future, the risk-adjusted present value factor may be approximated by $\left[\frac{1-(ER_m - R_f)\beta_g/(1-\phi)}{1+R_f}\right]^T$. Therefore, the present value of a single dividend D_T, for large T, may be approximated as follows:

$$V_0 \approx D_0 e^{\left[\bar{g} + \frac{1}{2}\frac{\sigma_\varepsilon^2}{(1-\phi)^2} - \left(R_f + (ER_m - R_f)\frac{\beta_g}{(1-\phi)}\right)\right]T} \tag{9}$$

This value will converge to zero, and thereby the sum of the present values of all future dividends (i.e., the firm value) will be finite, provided the expression inside the square brackets is negative. Thus, the restriction on long term growth is:

$$\bar{g} \le R_f + (ER_m - R_f)\frac{\beta_g}{(1-\phi)} - \frac{1}{2}\frac{\sigma_\varepsilon^2}{(1-\phi)^2} \tag{10}$$

The right hand side of this expression is similar to the CAPM risk-adjusted return with two modifications. First, risk is measured by the growth rate beta adjusted by the degree of predictability in the dividend stream, and second, since the growth rate is continuously compounded, we need to subtract one half the long-run variance of dividend shocks. Clearly this condition is less restrictive than for constant growth (i.e., $g < r$); and, more importantly, it imposes no restrictions on the short term dividend growth rate.

Figure 1 illustrates the upper bound condition described by Equation (10). We vary the level of mean reversion, ϕ, on the horizontal axis from 0.0 (strong) to 0.6 (weak), and include three levels of β_g : 0.5, 1.0, and 1.5. To set the riskless rate and the market risk premium, we use data from Professor Ken French's website. We have $R_f = 0.037$ and $E(R_m - R_f) = 0.0804$, which correspond to the sample averages from 1927 thru 2010. Last, σ_ε^2 is set arbitrarily at 0.01 because its exact value has a marginal effect on the upper bound.

The horizontal line at 3.7% corresponds to the constant growth Gordon model; clearly this is a low bar for the long run growth rate. The more realistic cases reflect varying degrees of mean reversion. In the strongest case, where ϕ = 0.0, the upper bound increases with beta. When β_g = 0.5, the risk-adjusted upper bound is 7.2%, and increases to 11.2% for β_g = 1.0, and to 15.3% for β_g = 1.5. Figure 1 shows that these upper bounds increase even faster as mean reversion weakens. No firm can be expected to grow permanently at such high rates.

5. A modified St. Petersburg game for stochastic growth stocks

As Durand (1957) shows, the classical St. Petersburg game (with some modifications) can describe an investment in a stock with a constant dividend growth rate. To capture the more realistic pattern of growth firms (as outlined in the previous section), we present a modified St. Petersburg game that is analogous to a stock with a mean reverting dividend growth rate in the equilibrium setting of the CAPM.

Consider the St. Petersburg game, where instead of a constant probability that the coin will fall on 'head', the probability is different for every toss; specifically, the probability that 'head' appears at the j^{th} toss is of the form $(a_j - 1)/a_j$, where $a_j > 1 \ \forall j$. Assume further that the player earns a different amount of dollars every toss, as long as the game continues: D_1 if the first toss is 'tail', D_2 if the second toss is 'tail', D_3 if the third toss is 'tail', and so on. That is, if the game lasts for *n* tosses, the player will earn $\sum_{j=1}^{n-1} D_j$ dollars. The expected payoff of the game is:

$$E = \sum_{n=1}^{\infty} D_n \sum_{j=n+1}^{\infty} \frac{a_j - 1}{\prod_{k=1}^{j} a_k} = \sum_{n=1}^{\infty} D_n \prod_{j=1}^{n} \frac{1}{a_j} \tag{11}$$

Assume now that the stream of payoffs, as the stream of dividends discussed above, grows at a mean reverting growth rate g_{t+1} (Equation 6). Note that the expected payoff for the n^{th} toss, $E_0 D_n$, is equivalent to the one appearing in Equation (8). Next, define the ratio $a_j = \frac{(1+R_f)}{1-(ER_m - R_f)z_j\beta_g}$; since the risk-free rate, R_f, and the market risk-premium, $(ER_m - R_f)z_j\beta_g$, are both positive, a_j is greater than 1. This guarantees a probability of 'head' between 0 and 1. Therefore, the expected payoff of the game is:

$$E = \sum_{n=1}^{\infty} \frac{(E_0 D_n) \prod_{j=1}^{n} \left[1-(ER_m - R_f)z_j\beta_g\right]}{\left(1+R_f\right)^n} \tag{12}$$

which matches exactly the stock value in the CAPM world (Equation 8).

The condition under which the value of the game is finite, therefore, is identical to the one that makes the stock price finite, as given in Equation (10). Since this condition is less restrictive than the classical St. Petersburg game (as discussed above), it provides an indirect solution to the paradox. That is, under the more realistic setup of a stochastic mean reverting growth rate, it is more likely that the value of the game is finite, and therefore the game fee the players are willing to pay is finite.

6. Conclusions

The St. Petersburg paradox describes a simple game of chance with infinite expected payoff, and yet any reasonable investor will pay no more than a few dollars to participate in the game. Researchers throughout history have provided a number of solutions as well as variations of the original paradox. One of these, developed by Durand (1957), shows that the standard St. Petersburg game can describe an investment in a firm with a constant growth rate of dividends.

To capture a more realistic growth pattern, we present a model that allows mean reversion in dividends. We then derive the risk adjustment required in a CAPM environment, and propose an equivalent St. Petersburg game. We show that the expected payoff of the modified game (or equivalently, the value of growth firms) is driven mainly by the long-run growth rate of the payoffs (dividends), while the short-term growth rate has a minor effect on the properties of the game or the firm. The model further shows that the condition under which the value of the game or the firm is finite is much less restrictive than that of the classical St. Petersburg game, and this might provide an indirect solution to the paradox.

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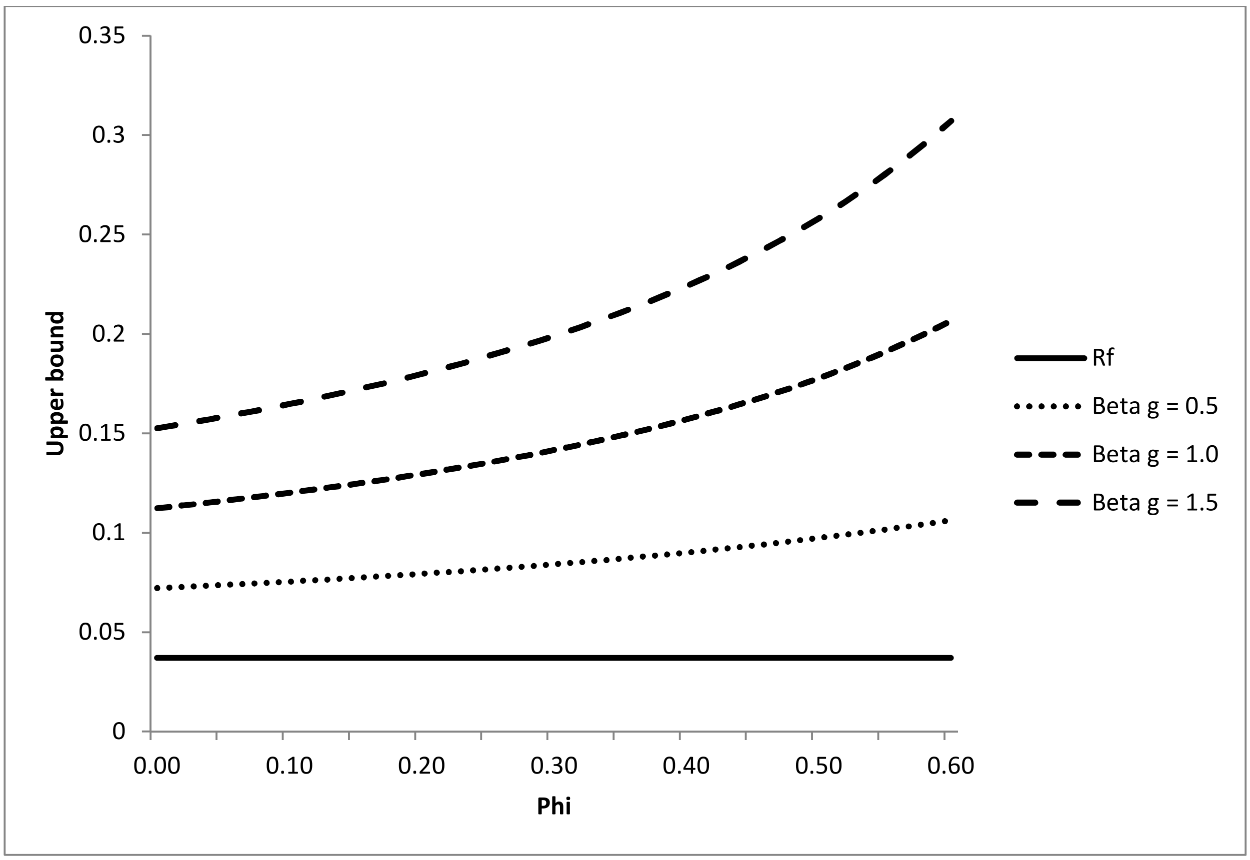
Figure 1. Upper bound on long run dividend growth rate

The three sloping lines represent the upper bound on the long run dividend growth rate, computed from Equation (10), as a function of the degree of mean reversion (ϕ), for three levels of β_g. The model parameters are: $R_f = 0.037$, market risk premium $E(R_m - R_f) = 0.0804$, and $\sigma_\varepsilon^2 = 0.01$. The horizontal line, set at $R_f = 0.037$, represents the upper bound on the constant (deterministic) growth rate in the Gordon model.



Appendix: *Proof of Equation (8)*

The sequence of future growth rates $G^{'} \equiv (g_1,\ g_2, \ldots, g_T)$ may be represented in matrix form as follows: $\Phi G = (1-\phi)\bar{g}i + G_0 + \mathrm{E}$, where the $T \times T$ matrix Φ consists of 1s along the main diagonal, $-\phi$ in the each of the cells immediately below the main diagonal and 0 everywhere else. i is a $T \times 1$ vector of 1s, G_0 is a column vector with ϕg_0 in the first row and 0 in the remaining rows, and $E^{'} \equiv (\varepsilon_1,\ \varepsilon_2, \ldots,\ \varepsilon_T)$ is a vector of random innovation terms. Using this set up, $\sum_{t=1}^{T} g_t$ has conditional mean $(1-\phi)\bar{g}\, i^{'}\Phi^{-1} i + i^{'}\Phi^{-1} G_0$ and variance $\sigma_\varepsilon^2\, i^{'}\Phi^{-1}(\Phi^{-1})^{'} i$. Using a result from time series analysis, we show next that these moments may be computed without inverting the Φ matrix. Define the vector $Z^{'} \equiv (z_T, z_{T-1}, \ldots, z_1) = i^{'}\Phi^{-1}$ and note that each element may be computed recursively from the previous one: $z_j = \phi z_{j-1} + 1$ for j=1, 2, ... , T, and starting value z_0=0. Hence, the expected future dividend is given by: $E_0 D_T = D_0 \exp\left[(1-\phi)\,\bar{g} \sum_{j=1}^{T} z_j + z_T \phi\, g_0 + (\sigma_\varepsilon^2/2) \sum_{j=1}^{T} z_j^2\right]$. Next, we derive the present value of each future dividend D_T starting from T=1, 2, and so on. Let $R = \frac{D_1}{V_0} - 1$ be the rate of return on a claim that pays off a single cash flow $\$D_1$ at T=1. Plug this return into the security market line (Equation 7) to show that the present value of D_1 is given by: $V_0 = (E_0 D_1)\left[\frac{1-(ER_m - R_f)Cov(D_1/(E_0 D_1),\ R_{m1})/\sigma_m^2}{1+R_f}\right]$. Using Stein's lemma it follows that: $Cov\left(\frac{D_1}{E_0 D_1},\ R_{m1}\right) = Cov(\varepsilon_1,\ R_{m1})$. Define the growth rate beta $\beta_g = Cov(\varepsilon_1,\ R_{m1})/\sigma_\varepsilon^2$.

Therefore, the present value of the first dividend is given by

$$V_0 = D_0\, e^{(1-\phi)\bar{g}\, z_1 + z_1 \phi\, g_0 + (\sigma_\varepsilon^2/2) z_1^2} \left[\frac{1-(ER_m - R_f) z_1 \beta_g}{1+R_f} \right].$$

Next, let V_1 be the time-1 value of a single cash flow $\$D_2$ expected at T=2. Again, let $R = \frac{V_1}{V_0} - 1$ be the rate of return (from 0 to 1) from holding the claim on $\$D_2$. Equation (7) implies that $V_0 = (E_0 V_1) \left[\frac{1-(ER_m - R_f) Cov(V_1/(E_0 V_1),\, R_{m1})/\sigma_m^2}{1+R_f} \right]$. Using a similar argument as in Fama (1977), we can show that $E_0 V_1 = (E_0 D_2) \left[\frac{1-(ER_m - R_f) z_1 \beta_g}{1+R_f} \right]$. Moreover, the ratio of V_1 to its conditional expectation one period prior, $E_0 V_1$, equals the ratio of cash flow expectations: $\frac{V_1}{E_0 V_1} = \frac{E_1 D_2}{E_0 D_2}$. Then, from Stein's lemma we have $Cov\left(\frac{V_1}{E_0 V_1},\, R_{m,1} \right) = Cov(z_2 \varepsilon_1 + z_1 \varepsilon_2,\, R_{m,1}) =$ $z_2\, Cov(\varepsilon_1, R_{m,1})$. Therefore, the present value of the second dividend is given by:

$$V_0 = D_0\, e^{(1-\phi)\bar{g} \sum_{j=1}^{2} z_j + z_2 \phi\, g_0 + (\sigma_\varepsilon^2/2) \sum_{j=1}^{2} z_j^2} \left[\frac{1-(ER_m - R_f) z_1 \beta_g}{1+R_f} \right] \left[\frac{1-(ER_m - R_f) z_2 \beta_g}{1+R_f} \right]$$

Proceeding in this fashion one may show that for any D_T, the present value is:

$V_0 = \frac{(E_0 D_T) \prod_{j=1}^{T} \left[1-(ER_m - R_f) z_j \beta_g \right]}{(1+R_f)^T}$. Thus, Equation (8) holds by the principle of value additivity. ■