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# Pricing assets with stochastic cash-flow growth

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# Pricing assets with stochastic cash-flow growth

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We model the time series behavior of dividend growth rates, as well as the profitability rate, with a variety of autoregressive moving-average processes, and use the capital asset pricing model (CAPM) to derive the appropriate discount rate. One of the most important implications of this research is that the rate of return beta changes with the time to maturity of the expected cash flow, and the degree of mean reversion displayed by the growth rate. We explore the consequences of this observation for three different strands of the literature. The first is for the value premium anomaly, the second for stock valuation and learning about long-run profitability, and the third is for the St. Petersburg paradox. One of the most surprising results is that the CAPM implies a higher rate of return beta for value stocks than growth stocks. Therefore, value stocks must have higher expected returns, and this is what is required theoretically in order to explain the well-known value premium anomaly.

*Keywords*: Asset pricing; Dividends; St. Petersburg paradox; Time series analysis

*JEL Classification*: G12, G17



## 1. Introduction

The constant growth dividend discount model is a simple and popular tool for approximating the intrinsic value of an asset. The appropriate discount rate is, quite often, taken from the capital asset pricing model (CAPM) of Sharpe (1964) and Lintner (1965). Thus, both the growth rate and the cost of capital are assumed to be constant parameters. The theoretical justification for this methodology may be found in the works of Gordon (1962) and Fama (1977).

However, within the context of the CAPM, it is easy to show that a constant discount rate is appropriate only when cash flows follow a random walk. Economic intuition, plus historical experience, suggests that the high cash flow growth rates experienced by many young firms (e.g., firms in the high-tech industry, such as IBM, Microsoft, and now Google) are unsustainable in the long run. Competition from new start-up companies invariably forces expected growth rates to decline over time. Hence, the short-run rate is typically much greater than the expected long-run profitability rate.

In the first part of this paper we model the time series behavior of dividend growth rates with a first-order autoregressive process, and then use the CAPM to derive the appropriate discount rate. One of the most important implications of this research is that the rate of return beta changes with the time to maturity of the expected cash flow and the degree of mean reversion displayed by the growth rates. The CAPM then implies that dividends received at different dates cannot have the same expected returm.

We explore the consequences of this observation for three different strands of the literature. For the first, consider that one of the most notable violations of the capital asset pricing model is the value premium anomaly. Empirically, value stocks have lower betas than growth stocks, yet growth stocks display lower average returns. Lettau and Wachter (2007, henceforth L&W) develop a stochastic discount factor model where only dividend risk is priced.† The model delivers higher expected returns for value firms because their cash flows occur in the near future, and covary more with shocks to aggregate dividends. Investors fear these shocks the most, and require a larger risk premium to buy and hold value firms.

The structural model for the aggregate dividend growth rate introduced by L&W is equivalent to a first-order

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†Koijen *et al*. (2010) propose a three-factor model that prices the cross-section of stocks and bonds and in addition provides an alternative explanation of the value premium.

autoregressive moving-average time series process. Assuming that shocks to the growth rate have constant covariance with the market return, we define the growth rate beta as this covariance divided by the market's rate of return variance. Then, applying the CAPM recursively in the fashion of dynamic programming, we derive closed-form solutions for the rate of return beta and the corresponding risk premium. It is surprising to find that growth stocks, which consist of long duration cash flows, have lower rate of return beta than value stocks, which are made up of short duration cash flows. Consistent with the empirically observed value premium, the equilibrium expected return for growth stocks must be lower than the return for value stocks. Therefore, the CAPM may explain the well-known value premium – provided that the rate of return beta is properly adjusted for mean reversion in the expected future dividends.

The foundation for this result is the assumption that dividend growth follows a normal process with time-varying mean. The corresponding reduced form time series model is ARMA(1, 1), and it may display positive or negative serial correlation. If shocks to the growth rate display positive correlation, cash flows are expected to grow faster (slower) when the market return is high (low), and this will increase the rate of return beta. In turn, it follows that longer duration cash flows will display higher expected returns. To illustrate this result with numerical examples, we calibrate the model using the same macro parameters as those reported by L&W (see Section 3.1 for specific details), and find that the rate of return beta and return volatility are increasing functions of time to maturity. Of course, these results are inconsistent with the value premium.

The surprise finding is that when we model negative serial correlation in dividend growth, the rate of return beta, the risk premium, and the return volatility fall as the time horizon – before a cash flow is received, increases.† To see why, note that if the current growth rate is above its long-run mean, then negative autocorrelation implies that cash flows are expected to grow more slowly in the future. In turn, this leads to lower correlation with the overall market. From this point of view, long duration cash flows may be less risky than short ones because growth will display a tendency to reverse itself, and this acts as a hedge. The implication of these results is that the CAPM has the potential to explain the value premium anomaly.

The second application of this research is to asset valuation when the long-run profitability rate is unknown. In particular, assets that appear to be irrationally overpriced may actually be fairly priced if one takes into account uncertainty about expected future profitability. Pastor and Veronesi (2003, 2006; henceforth P&V) develop several versions of the Gordon dividend discount model in continuous time, and show that higher uncertainty about long-run profitability leads to higher market-to-book ratio.

It turns out that similar results hold within a CAPM world. We model the accounting rate of return on equity with a first-order autoregressive process. To account for the business cycle, the expected long-run profitability rate is assumed also to follow a mean reverting process. Thus, investors do not know how profitable a firm might be in the long run but they rationally learn about its potential value by observing the current rate. We then use CAPM to prove that the market-to-book ratio is a convex function of the long-run profitability rate. The intuition for this result comes from the fact that when uncertainty is high, a sequence of high growth rates has a much bigger impact on future expected returns than a run of low growth rates. Therefore, both the expected future firm value and the current value increase.

The third and last application of our model is to the classical St. Petersburg paradox. Durand (1957) shows that the St. Petersburg game can be used to describe a conventional model of stock prices. In particular, the analogy is based on the assumption that the firm's expected future dividends (as the game's future payoffs) grow at a constant rate. If dividends are discounted at a constant rate, this rate must be greater than the dividend growth rate to assure a finite stock value. The value of the stock is infinite if dividends grow at an equal or higher rate than the discount rate.

The paradox arises for two reasons. First, the assumption that the dividend stream will grow at a constant rate permanently is unrealistic. As discussed above, competition will eliminate abnormal profits after a period of time. In the long run, new market entrants will force the earnings growth rate to slow down to a level consistent with the growth rate of the overall economy. The second reason is that the degree of risk implicit in the earnings growth stream may cause investors to change the risk-adjusted discount rate, i.e. the probability of actually receiving the expected dividend stream. We formalize these ideas within the context of the CAPM. We model the dividend growth rate as an autoregressive process so that the current rate can be very large but the long-run growth rate is closer to that of the economy. We then use the CAPM to derive the appropriate risk-adjusted discount rate. We show that the value of the stock can be finite without the unreasonable condition on the constant growth rate.

The remaining of the paper is organized as follows. Section 2 introduces the general time series model for the dividend growth rate and the valuation based on the CAPM. The following section shows that, by the law of one price, the model holds in an arbitrage-free economy. Section 3 presents three applications of the model. The first is for the value premium anomaly, the second for stock valuation and learning about long-run profitability, and the third is for the St. Petersburg paradox. Section 4 concludes the paper. All proofs are in Appendix B.

†This result is based on the same level of negative correlation as that reported by L&W.

## 2. Stochastic dividend growth models

In the first subsection, we model the dividend growth rate as an autoregressive process so that the current rate can be very large but the long-run growth rate is expected to be much lower and therefore closer to that of the macroeconomy. We then use the CAPM to derive the present value of an asset whose dividends may be modeled as an autoregressive process. In Section 2.2 we show that the roots of our model go back to Rubenstein's (1976) paper on the valuation of uncertain income streams. We use state preference theory to show that our model is consistent with arbitrage-free asset pricing.

### *2.1. Valuation of assets with the capital asset pricing model*

Let $D_t$ be the time-$t$ value of dividends or earnings (for simplicity we will use these two terms interchangeably). We assume that dividends grow at a mean reverting growth rate $g_{t+1}$: $D_{t+1} = (e^{g_{t+1}})D_t$. To model mean reversion, we assume that the continuously compounded growth rate follows a first-order autoregressive process AR(1):

$$g_{t+1} = (1-\phi_1)\bar{g} + \phi_1 g_t + \varepsilon_{t+1}, \tag{1}$$

where $\bar{g}$ is the long-run (unconditional) mean dividend growth rate, and $\phi_1$ is the autoregressive coefficient. We make the usual assumptions to insure that the process is stationary and the growth rate is mean reverting. The innovation terms $\varepsilon_{t+1}$ are normally distributed random variables with mean zero, variance $\sigma_\varepsilon^2$, no serial correlation, and constant covariance with the market portfolio. The major implication of the AR model is that while the current growth rate can be abnormally large, in the long-run earnings growth should slow down to a lower rate $\bar{g}$. Intuitively, we expect $\bar{g}$ to be close to the growth rate for the overall economy because of competitive pressures brought about by new startup companies.

To apply the dividend discount model we need to forecast the path of expected future dividends using equation (1), and also take into account market risk using the CAPM. The conditional expectation (assuming $t=0$) of a future dividend $D_T = D_0\, e^{\sum_{t=1}^{T} g_t}$ has a rather simple form (Appendix A shows the derivation for a general $p$-order autoregressive process). To compute this expectation, we define the auxiliary autoregressive vector $Z' \equiv (z_T, z_{T-1}, \ldots, z_1)$, where each element may be computed recursively from the previous one: $z_j = \phi_1 z_{j-1} + 1$ for $j = 1, 2, \ldots, T$, and starting value of $z_0 = 0$. Let $g_0$ be the current growth rate; then, conditional on time 0 information, the expected time-$T$ dividend can be computed as follows:

$$E_0 D_T = D_0\, e^{(1-\phi_1)\bar{g}(\sum_{j=1}^{T} z_j) + z_T\phi_1 g_0 + (\sigma_\varepsilon^2/2)\sum_{j=1}^{T} z_j^2}. \tag{2}$$

Equation (2) shows that the expected future dividend is a convex function of the current and long-run growth rate. However, the long-run rate has a bigger impact because of the larger multiplier due to the cumulative effect of serial correlation.

To obtain the risk-adjusted present value of each future dividend we use the CAPM of Sharpe (1964) and Lintner (1965):

$$ER = R_f + [ER_m - R_f]\beta_{ROR}, \tag{3}$$

where $ER$ is the single period return on the asset, $R_f$ is the risk-free rate of interest, $ER_m$ is the expected market portfolio return, and market risk is measured by the rate of return beta ($\beta_{ROR}$). The following proposition shows how to discount expected future dividends.

**Proposition 1:** *The present value of a single future dividend $D_T$ is given by its conditional expected future value adjusted for its market risk and discounted to the present at the risk-free rate of interest:*

$$V_0 = \frac{(E_0\, D_T)\prod_{j=1}^{T}[1-(ER_m - R_f)z_j\beta_g]}{(1+R_f)^T}, \tag{4A}$$

*where the growth rate beta, $\beta_g$, is defined as the covariance between the growth rate innovation ($\varepsilon_{t+1}$) and the market return, divided by the variance of the market return. Summing over all future expected dividends yields the CAPM price for an asset with a stochastic growth rate:*

$$P_0 = \sum_{t=1}^{\infty}\frac{(E_0\, D_t)\prod_{j=1}^{t}[1-(ER_m - R_f)z_j\beta_g]}{(1+R_f)^{\,t}}. \tag{4B}$$

**Proof:** Appendix B provides the proof for a general AR($p$) process.

Equations (4A) and (4B) may seem a little odd because the expectation is taken with respect to the physical (i.e. real) probability measure, and at the same time, the riskless rate is used to discount future cash flows. To provide some intuition, we rewrite (4 A) in the more familiar textbook formula. That is, discount the expected cash flow with a risk-adjusted cost of capital: $V_0 = E_0\, D_T/(1+ER_T)^T$, where the discount rate is given by

$$E\,R_T = \frac{1+R_f}{\left(\prod_{j=1}^{T}[1-(ER_m - R_f)z_j\beta_g]\right)^{1/T}} - 1.$$

We note that the cost of capital will, in general, depend on the cash flow maturity as well as the degree of mean reversion in the growth rate.

Our pricing model retains the characteristics one would expect to observe in a risk-averse environment. For example, price is inversely related to the market risk premium, growth rate beta, and risk-free rate. Price is also expected to increase with the current and long-run dividend growth rate. Higher growth rate volatility leads to higher prices because long periods of above average growth have a bigger impact on present value than periods of low growth.

Equation (4B) neatly captures both the degree of dividend predictability and the corresponding risk adjustment. For example, as $\phi_1$ increases from 0 to 1 the strength of mean reversion decreases and dividends become less predictable. Then, the penalty for risk, implied by the certainty equivalent formula, increases in proportion to the growth rate beta. Alternatively stated, more persistent shocks are riskier because they have a larger impact on current price. This implies that stock returns are more sensitive to dividend growth shocks and thus become riskier.†

We note also that Gordon's deterministic growth model is a special case of equation (4B); it is obtained by setting $\phi_1 = 0$ and $\sigma_\varepsilon^2 = 0$. In this case, dividends are expected to grow in a deterministic fashion at a constant rate $\bar{g}$. The beta factor ($\beta_g$) equals zero and the annual discount rate $r$ equals the risk-free rate $R_f$.

A simple generalization of Gordon's model is one with stochastic growth but no serial correlation: $g_{t+1} = \bar{g} + \varepsilon_{t+1}$. In this case, (log) earnings or dividends follow a random walk with drift, and the stock price has a closed-form solution similar to Durand's (1957) formula:

$$P_0 = \frac{D_0 e^{g*}}{r - e^{g*}}, \tag{5}$$

where the discount rate $r$ is given by

$$\frac{(1 + R_f)}{1 - (ER_m - R_f)\beta_g},$$

and the adjusted growth rate is $g^* = \bar{g} + \sigma_\varepsilon^2/2$. The major drawback of these two models, however, is that unlike our autoregressive model, they do not allow a distinction between current growth – which can be abnormally high, and long-run growth.

### 2.2. General valuation of stochastic dividend growth

The roots of our growth valuation model go back to Rubenstein's (1976) paper on the valuation of uncertain income streams and the pricing of options. In fact, our dividend discount model generalizes Rubenstein's theorem 2 to the case where the growth rate is stationary but not serially independent. In this section we discuss the connection to Rubenstein's dividend growth model and, in the process, show that our methodology is consistent with a one factor arbitrage-free asset pricing model.

Let $\aleph = \{\aleph_1, \aleph_2, \ldots, \aleph_S\}$ be a complete list of all states of nature as of time $t$, where all states are assumed to be mutually exclusive. We associate a strictly-positive probability $p_s$ with each state of nature, and thus $\sum_{s=1}^{S} p_s = 1$. In the context of complete markets there exists an Arrow–Debreu security that pays \$1 in state $s$, and zero in all other states; let $\pi_s$ be the price of such a security.

A random cash flow $D$ pays off $\$D_s$ if state $s$ occurs at time $t$ (formally, we identify $D$ as a mapping from $\aleph$ to the real line. $D$: $\aleph \to \mathrm{R}^1$). From State Preference Theory, the time $t-1$ market value of $D$ is given by the sum, over all states of nature,

$$V_{t-1} = \sum_{s=1}^{S} \pi_s D_s = \sum_{s=1}^{S} p_s D_s \Lambda_s = E(D\Lambda), \tag{6}$$

where the random variable $\Lambda_s = \pi_s/p_s$ is commonly referred to as the pricing kernel or the stochastic discount factor (Cochrane 2005).

Equation (6) is the central contribution of modern asset pricing theory: the asset price is given by the expected value (across states of nature) of the market discount factor times the cash flow. The risk adjustment arises from co-variation between the cash flow and the random discount factor. Indeed, to make this result more intuitive, one may use the covariance representation:

$$V_{t-1} = \frac{E(D)[1 + \mathrm{Cov}(D/E(D), \Lambda/E(\Lambda))]}{1 + R_f}, \tag{7}$$

where $E(\Lambda) = 1/(1 + R_f)$.

But it is easily shown that equation (6) also implies the existence of the CAPM. To this end, let $R$ be the single-period rate of return, then $1 = E(\Lambda(1 + R))$. This relation implies that the excess return (over the riskless rate) must be orthogonal to the discount factor: $0 = E(\Lambda(R - R_f))$. Thus, the risk premium for any given security or portfolio – including the market portfolio, equals minus the covariance between the return and the scaled factor: $E(R) - R_f = -\mathrm{Cov}(\Lambda/E\Lambda,\ R)$ and $E(R_m) - R_f = -\mathrm{Cov}(\Lambda/E\Lambda,\ R_m)$. If the SDF is an affine function of the market return, then the ratio of these two equations yields:

$$\frac{E(R) - R_f}{E(R_m) - R_f} = \frac{\mathrm{Cov}(R, R_m)}{\sigma_m^2}$$

and the CAPM follows immediately. This equation yields an analogous representation of equation (7):

$$V_{t-1} = \frac{(ED)(1 - (ER_m - R_f)\beta_g)}{1 + R_f}. \tag{8}$$

Proposition 1 is based on equation (8), but one could just as easily begin with a stochastic discount factor representation. Theorem 2 of Rubenstein (1976) extends these results to a multi-period setting. In particular, consider the cash flow sequence $\{D_t\}_{t=1}^{\infty}$ with a (continuously compounded) stochastic growth rate $g_{t+1} = \bar{g} + \varepsilon_{t+1}$. Assume also that the dividend growth rate and the factor return are serially uncorrelated, and their respective lagged values are uncorrelated with each other, then theorem 2 yields a closed form solution analogous to our equation (5).

## 3. Applications of the stock valuation model

In the first subsection we model the dividend growth rate as an autoregressive moving-average process and use the CAPM to explain the value premium phenomenon. In the second subsection, we explore asset valuation when investors do not directly observe the long-run profitability rate, but learn about this rate by observing the current rate. The third deals with the St. Petersburg paradox and

†We are grateful to an anonymous referee for comments on this result.

derives a new, less stringent, condition on the dividend growth rate.

### *3.1. CAPM analysis of the value premium*

One of the most notorious violations of the CAPM is the so-called "Value Premium". Almost 80 years ago Graham and Dodd (1934) observed that growth stocks experience lower average returns than value stocks. By definition, growth stocks are characterized by high price to fundamentals ratio (such as high price to earnings ratio), whereas value stocks display low price to fundamentals ratio. After the development of the CAPM by Sharpe (1964) and Lintner (1965), a simple explanation for the value premium may be that higher systematic risk requires a higher risk premium. If value stocks have relatively higher beta risk than growth stocks, then firms with a high dividend price ratio must offer a higher risk-adjusted average return.

But the empirical evidence consistently rejects this explanation. Fama and French (1992) and Lettau and Wachter (2007) are but two examples from a long list in the literature to show that value stocks have lower systematic risk and higher average returns than growth stocks. Contrary to the prediction of the CAPM, the empirical slope of the security marker line appears to be negative.

Lettau and Wachter (L&W) propose a stochastic dividend growth model that explains why value stocks should have higher expected returns than growth stocks. The key for their result is a dynamic stochastic discount factor where the price of risk is correlated with the growth rate in aggregate dividends. Stated alternatively, in their model dividend shocks are priced, whereas discount rate shocks are not. Their model has a total of 12 free parameters; an additional 5 parameters from the variance-covariance matrix are set at zero. Model parameters are calibrated to fit more than one hundred years of aggregate data obtained from Campbell (1999).

The major implication of their model is that the risk premium and rate of return volatility should be inversely related with time to maturity. Thus, stocks or portfolios heavily tilted toward low duration cash flows (i.e. value stocks) should be characterized by higher expected returns, lower return variance, and higher Sharpe ratios than portfolios of high duration assets. L&W use simulated data from their model and find that value stocks indeed have higher excess returns and higher Sharpe ratios than growth portfolios. A quite remarkable result from their analysis is that the risk premium for a claim on a dividend two years from now is 18% per year and only 4% per year for a claim on a dividend expected 40 years from now.

It is well known that betas are nonmonotonic with time to maturity, therefore, the traditional CAPM cannot explain the additional risk premium required by value stocks. On the other hand, the stochastic dividend discount model developed in Section 2 suggests that the risk premium may change with the maturity of the cash flow, and may also depend on the degree of mean reversion displayed by the dividend growth rate. Therefore, it is natural to ask whether a version of the CAPM that accounts for cash flow characteristics may be capable of explaining the value premium. In this section we explore this possibility.

In order to expand the class of growth rate processes one may encounter in practice, we model $g_{t+1}$ as a first-order autoregressive moving-average process ARMA(1,1):

$$g_{t+1} = (1-\phi_1)\bar{g} + \phi_1 g_t + \varepsilon_{t+1} - \theta_1 \varepsilon_t, \tag{9}$$

where $(\phi_1, \theta_1)$ are, respectively, the autoregressive and moving-average coefficients. Again, we assume the process is stationary and shocks to the growth rate have constant beta (denoted by $\beta_g$), that is constant covariance with the market return. The ARMA model may display mean reversion (negative serial correlation) or momentum (positive correlation). It can be easily shown that the structural model for (log) dividend growth rate assumed by L&W is analogous to our ARMA model.

The following proposition shows how to compute asset prices that are consistent with the CAPM.

**Proposition 2:** *Suppose the CAPM holds and the dividend growth rate follows the ARMA(1,1) process. Define also the sequence of multipliers $w_j$ to account for the serial correlation in the cumulative growth rate. The sequence of $w_j$'s can be computed recursively from two auxiliary equations: $w_j = z_j - \theta_1 z_{j-1}$ and $z_j = \phi_1 z_{j-1} + 1$ for $j = 1, 2, \ldots, T$; the starting value is $z_0 = 0$. Then, the price of zero-coupon equity that pays a single future dividend $D_{t+T}$ is given by*

$$V_{T,\,t} = \frac{(E_t\, D_{t+T}) \prod_{j=1}^{T} [1 - (ER_{\mathrm{m}} - R_{\mathrm{f}}) w_j \beta_{\mathrm{g}}]}{(1+R_{\mathrm{f}})^{T}}, \tag{10A}$$

*where the expected future cash flow is*

$$E_t D_{t+T} = D_t e^{(\phi_1 g_t - \theta_1 \varepsilon_t) z_T + (1-\phi_1)\bar{g} \sum_{j=1}^{T} z_j + (\sigma_\varepsilon^2/2) \sum_{j=1}^{T} w_j^2}. \tag{10B}$$

For values of $\phi_1$ and $\theta_1$ between 0 and 1, both $z_j$ and $w_j$ increase with $T$, therefore the current price increases with the current growth rate and the long-run growth rate. The price dividend ratio increases also with the growth rate volatility. This result is due to the convexity of compounded cash flows: a string of high growth rates has a larger impact on future dividends than a run of low rates.

We show next that this proposition leads to closed form solutions for the rate of return beta and the Sharpe ratio for zero coupon equity. Thus, let $R_{T,t+1}$ be the holding period return from holding zero-coupon equity from $t$ to $t+1$: $R_{T,t+1} = V_{T-1,t+1}/V_{T,t}$, where both the current and next period price are obtained from equation (10A). It can be easily shown that the gross rate of return is a function of the growth rate shock adjusted for risk:

$$1 + R_{T,t+1} = e^{w_T \varepsilon_{t+1} - (\sigma_\varepsilon^2/2) w_T^2} \left( \frac{1 + R_{\mathrm{f}}}{1 - (ER_{\mathrm{m}} - R_{\mathrm{f}}) w_T \beta_{\mathrm{g}}} \right).$$

Table 1. Characteristics of zero-coupon equity when the dividend growth rate displays positive serial correlation (momentum). This table reports values of $(z_j, w_j)$, price-dividend ratio, rate of return beta ($\beta_{\mathrm{ROR}}$), market risk premium $E(R_{T,t+1} - R_{\mathrm{F}})$, return volatility, and Sharpe ratio for zero-coupon equity. The sequence $(z_j,\ w_j)$ are computed recursively from two auxiliary equations: $z_j = 0.545\, z_{j-1} + 1$, and $w_j = z_j - 0.16 z_{j-1}$ for $j = 1, 2, \ldots, T$, with starting value $z_0 = 0$..

| Time horizon | $Z$ | $W$ | Price–dividend ratio | $\beta_{\mathrm{ROR}}$ | Risk premium | Volatility | Sharpe ratio |
|---|---|---|---|---|---|---|---|
| 1 | 1.00 | 1.00 | 0.976 | 0.53 | 0.033 | 0.08 | 0.22 |
| 2 | 1.55 | 1.39 | 0.955 | 0.74 | 0.046 | 0.15 | 0.22 |
| 3 | 1.84 | 1.59 | 0.937 | 0.86 | 0.054 | 0.20 | 0.21 |
| 4 | 2.00 | 1.71 | 0.920 | 0.92 | 0.058 | 0.23 | 0.21 |
| 5 | 2.09 | 1.77 | 0.905 | 0.96 | 0.060 | 0.25 | 0.21 |
| 6 | 2.14 | 1.81 | 0.891 | 0.98 | 0.061 | 0.26 | 0.21 |
| 7 | 2.17 | 1.82 | 0.877 | 0.99 | 0.062 | 0.27 | 0.21 |
| 8 | 2.18 | 1.83 | 0.863 | 0.99 | 0.062 | 0.27 | 0.21 |
| 9 | 2.19 | 1.84 | 0.850 | 1.00 | 0.063 | 0.27 | 0.21 |
| 10 | 2.19 | 1.84 | 0.837 | 1.00 | 0.063 | 0.27 | 0.21 |
| 11 | 2.20 | 1.84 | 0.825 | 1.00 | 0.063 | 0.27 | 0.21 |
| 12 | 2.20 | 1.85 | 0.812 | 1.00 | 0.063 | 0.27 | 0.21 |
| 13 | 2.20 | 1.85 | 0.800 | 1.00 | 0.063 | 0.27 | 0.21 |
| 14 | 2.20 | 1.85 | 0.788 | 1.00 | 0.063 | 0.27 | 0.21 |
| 15 | 2.20 | 1.85 | 0.776 | 1.00 | 0.063 | 0.27 | 0.21 |
| 16 | 2.20 | 1.85 | 0.764 | 1.00 | 0.063 | 0.27 | 0.21 |
| 17 | 2.20 | 1.85 | 0.752 | 1.00 | 0.063 | 0.27 | 0.21 |
| 18 | 2.20 | 1.85 | 0.741 | 1.00 | 0.063 | 0.27 | 0.21 |
| 19 | 2.20 | 1.85 | 0.730 | 1.00 | 0.063 | 0.27 | 0.21 |
| 20 | 2.20 | 1.85 | 0.719 | 1.00 | 0.063 | 0.27 | 0.21 |
| 21 | 2.20 | 1.85 | 0.708 | 1.00 | 0.063 | 0.27 | 0.21 |
| 22 | 2.20 | 1.85 | 0.697 | 1.00 | 0.063 | 0.27 | 0.21 |
| 23 | 2.20 | 1.85 | 0.687 | 1.00 | 0.063 | 0.27 | 0.21 |
| 24 | 2.20 | 1.85 | 0.676 | 1.00 | 0.063 | 0.27 | 0.21 |
| 25 | 2.20 | 1.85 | 0.666 | 1.00 | 0.063 | 0.27 | 0.21 |
| 26 | 2.20 | 1.85 | 0.656 | 1.00 | 0.063 | 0.27 | 0.21 |
| 27 | 2.20 | 1.85 | 0.646 | 1.00 | 0.063 | 0.27 | 0.21 |
| 28 | 2.20 | 1.85 | 0.636 | 1.00 | 0.063 | 0.27 | 0.21 |
| 29 | 2.20 | 1.85 | 0.627 | 1.00 | 0.063 | 0.27 | 0.21 |
| 30 | 2.20 | 1.85 | 0.617 | 1.00 | 0.063 | 0.27 | 0.21 |

The time series of returns are independent over time; however, they display heteroscedasticity. Positive (negative) shocks to the dividend growth rate increase (decrease) current returns. Last, the effect of the autoregressive moving-average parameters, captured by $w_j$, magnifies growth shocks and the growth rate beta.†

The following proposition shows how to compute the rate of return beta, return volatility, and Sharpe ratio for zero coupon equity.

**Proposition 3:** *Suppose the CAPM holds and the dividend growth rate follows the ARMA(1,1) process. Then, the rate of return beta, return variance, and the Sharpe ratio for the zero-coupon equity are given by*

$$\beta_{\mathrm{ROR},\,T} = \frac{(1+R_{\mathrm{f}})w_T\beta_{\mathrm{g}}}{1-(ER_{\mathrm{m}}-R_{\mathrm{f}})w_T\beta_{\mathrm{g}}}, \tag{11A}$$

$$\mathrm{Var}(R_{T,t+1}) = \left(\frac{1+R_{\mathrm{f}}}{1-(ER_{\mathrm{m}}-R_{\mathrm{f}})w_T\beta_{\mathrm{g}}}\right)^2 (e^{w_T^2\sigma_\varepsilon^2}-1), \tag{11B}$$

$$SR_T = \frac{(ER_{\mathrm{m}}-R_{\mathrm{f}})w_T\beta_{\mathrm{g}}}{[e^{w_T^2\sigma_\varepsilon^2}-1]^{1/2}}. \tag{11C}$$

It is not immediately clear from proposition 3 what type of relationship may exist between the stock return characteristics, such as rate of return beta or volatility, and cash flow duration. To shed some light on this point, we estimate the price-dividend ratio, market risk premium, rate of return standard deviation, and the Sharpe ratio for zero coupon equity with a single cash flow $D_{t+T}$.

Tables 1 and 2 report these values for time to maturity ($T$) ranging from 1 to 30 years. We use the same macro parameters as L&W's statistics. Thus, we set the annualized risk-free rate at 0.0193, the market risk premium $E(R_{\mathrm{M}} - R_{\mathrm{F}})$ at 0.0633, the long-run dividend growth rate at 0.028, and the growth rate variance at $0.1448^2$. In table 1, we set the growth rate beta at 0.5, while the autoregressive moving-average parameters are set at 0.545 and 0.16, respectively, to model positive growth rate autocorrelation. These values imply that the correlogram decays exponentially starting from a first-order correlation coefficient of 0.50.

†When applied to multi-period problems, the classical CAPM precludes stochastic variation in the parameters of the opportunity set. Also, the model precludes correlation between realized returns and the opportunity set (see Fama (1977) for a thorough discussion of these points).

Table 2. Characteristics of zero-coupon equity when the dividend growth rate displays negative serial correlation (mean reversion). This table reports values of $(z_j, w_j)$, price-dividend ratio, rate of return beta ($\beta_{\mathrm{ROR}}$), market risk premium $E(R_{T,t+1} - R_{\mathrm{F}})$, return volatility, and Sharpe ratio for zero-coupon equity. The sequence $(z_j, w_j)$ are computed recursively from two auxiliary equations: $z_j = 0.3z_{j-1} + 1$, and $w_j = z_j - 0.4z_{j-1}$ for $j = 1, 2, \ldots, T$, with starting value $z_0 = 0$..

| Time horizon | Z | $W$ | Price–dividend ratio | $\beta_{\mathrm{ROR}}$ | Risk premium | Volatility | Sharpe ratio |
|---|---|---|---|---|---|---|---|
| 1 | 1.00 | 1.00 | 0.912 | 1.38 | 0.087 | 0.20 | 0.54 |
| 2 | 1.30 | 0.90 | 0.855 | 1.23 | 0.078 | 0.16 | 0.54 |
| 3 | 1.39 | 0.87 | 0.808 | 1.19 | 0.075 | 0.15 | 0.54 |
| 4 | 1.42 | 0.86 | 0.765 | 1.18 | 0.074 | 0.15 | 0.54 |
| 5 | 1.43 | 0.86 | 0.725 | 1.17 | 0.074 | 0.15 | 0.54 |
| 6 | 1.43 | 0.86 | 0.688 | 1.17 | 0.074 | 0.15 | 0.54 |
| 7 | 1.43 | 0.86 | 0.652 | 1.17 | 0.074 | 0.15 | 0.54 |
| 8 | 1.43 | 0.86 | 0.618 | 1.17 | 0.074 | 0.15 | 0.54 |
| 9 | 1.43 | 0.86 | 0.586 | 1.17 | 0.074 | 0.15 | 0.54 |
| 10 | 1.43 | 0.86 | 0.556 | 1.17 | 0.074 | 0.15 | 0.54 |
| 11 | 1.43 | 0.86 | 0.527 | 1.17 | 0.074 | 0.15 | 0.54 |
| 12 | 1.43 | 0.86 | 0.499 | 1.17 | 0.074 | 0.15 | 0.54 |
| 13 | 1.43 | 0.86 | 0.474 | 1.17 | 0.074 | 0.15 | 0.54 |
| 14 | 1.43 | 0.86 | 0.449 | 1.17 | 0.074 | 0.15 | 0.54 |
| 15 | 1.43 | 0.86 | 0.426 | 1.17 | 0.074 | 0.15 | 0.54 |
| 16 | 1.43 | 0.86 | 0.404 | 1.17 | 0.074 | 0.15 | 0.54 |
| 17 | 1.43 | 0.86 | 0.383 | 1.17 | 0.074 | 0.15 | 0.54 |
| 18 | 1.43 | 0.86 | 0.363 | 1.17 | 0.074 | 0.15 | 0.54 |
| 19 | 1.43 | 0.86 | 0.344 | 1.17 | 0.074 | 0.15 | 0.54 |
| 20 | 1.43 | 0.86 | 0.326 | 1.17 | 0.074 | 0.15 | 0.54 |
| 21 | 1.43 | 0.86 | 0.309 | 1.17 | 0.074 | 0.15 | 0.54 |
| 22 | 1.43 | 0.86 | 0.293 | 1.17 | 0.074 | 0.15 | 0.54 |
| 23 | 1.43 | 0.86 | 0.278 | 1.17 | 0.074 | 0.15 | 0.54 |
| 24 | 1.43 | 0.86 | 0.263 | 1.17 | 0.074 | 0.15 | 0.54 |
| 25 | 1.43 | 0.86 | 0.250 | 1.17 | 0.074 | 0.15 | 0.54 |
| 26 | 1.43 | 0.86 | 0.237 | 1.17 | 0.074 | 0.15 | 0.54 |
| 27 | 1.43 | 0.86 | 0.225 | 1.17 | 0.074 | 0.15 | 0.54 |
| 28 | 1.43 | 0.86 | 0.213 | 1.17 | 0.074 | 0.15 | 0.54 |
| 29 | 1.43 | 0.86 | 0.202 | 1.17 | 0.074 | 0.15 | 0.54 |
| 30 | 1.43 | 0.86 | 0.191 | 1.17 | 0.074 | 0.15 | 0.54 |

The first two columns of table 1 show that the sequence $(z_j, w_j)$ increases quite rapidly with time to maturity. However, from $T = 11$ years and on, both values converge to a steady-state level; convergence to a constant value is required in order to achieve a finite price. Consistent with economic intuition, we expect the present value of $D_{t+T}$ to fall with time to maturity, and indeed the price dividend ratio in column 3 starts at 0.976 and decreases monotonically thereafter.

The rate of return beta, the risk premium, and the return volatility (shown in columns 5 thru 7) are directly dependent on $w_j$, therefore they must share the same time pattern. Beta starts at 0.53 for $T = 1$ and increases monotonically up to 1.00 for cash flows with at least 9 years to maturity. The risk premium ranges from 3.3% to 6.3%, while the equity return standard deviation increases rapidly from 8% to 27%. For all practical purposes, the Sharpe ratio is constant at 0.22. This last result suggests that our model with only seven parameters is not flexible enough to allow the risk premium to move independently of return volatility.

The intuition for the results displayed in table 1 is straightforward: growth rate shocks have positive serial correlation, thus they take longer to die down. Since these shocks are positively correlated with the market return, cash flows are expected to grow faster (slower) when the market return is high (low), and this will increase the rate of return beta. It is for this reason that longer duration cash flows will display higher systematic risk.

The range of values displayed in table 1 is not inconsistent with those derived by L&W for zero coupon equity. For example, the long-run Sharpe ratio converges to the same value in both our model and L&Ws. Return volatility and $\beta_{\mathrm{ROR}}$ display the same pattern – for values of $T$ up to 10 years, as the results in figure 5 of L&W.

Taken together the results in table 1 are inconsistent with the value premium. However, table 2 tells a different story. To model negative serial correlation in dividend growth, we set $\phi_1 = 0.30$ and $\theta_1 = 0.40$; these values imply a first-order serial correlation coefficient of $-0.10$. This degree of negative correlation is roughly the same as that found by L&W in one hundred years of data from 1890 to 2002 (table VI, p. 71). The growth rate beta is set to 1.25, and all other parameters are the same as in table 1.

The first two columns of table 2 show that while the sequence $z_j$ is still increasing, $w_j$ is decreasing with time to maturity. Thus, the rate of return beta, the risk premium, and the return volatility fall as the time horizon – before a cash flow is received, increases. The risk premium starts at 8.7% for a cash flow with one year to maturity and falls to 7.4% for a 30-year zero coupon equity. The rate of return beta falls from 1.38 to 1.17. The Sharpe ratio is again constant at 0.54 for all $T$.

Clearly, these results must be due to the negative serial correlation in dividend growth. To see why, note that if the current growth rate is above its long-run mean, then negative autocorrelation implies that cash flows are expected to grow more slowly in the future. In turn, this leads to lower correlation with the overall market. From this point of view, long duration cash flows may be less risky than short ones because growth will display a tendency to reverse itself: an extended period of high growth is likely to be followed by slower growth (and *vice versa*).

The most important implication of the results in table 2 is that the CAPM has the potential to explain the value premium anomaly. The market risk premium pattern in table 2 is strikingly similar to that reported in the top panel of figure 4 of L&W. In both cases, the risk premium declines with time to maturity $T$. Thus, firms with low duration cash flows – such as value firms, display higher rate of return beta, higher risk premium, and higher return volatility than firms with long duration cash flows – such as growth firms.

To establish whether these results hold in practice, empirical analysis must be based on a different type of sort. Firms should be sorted by degree of serial correlation in fundamental variables such earnings to price or cash flow to price. Also, higher-order autoregressive moving-average processes may be required to capture variation in the long-run growth rate driven by the business cycle.

### 3.2. *Stock valuation with a known profitability rate*

Between 1996 and 2000, stock prices of technology firms experienced phenomenal growth, but by October 2002 prices were back to the same starting point. A natural question to ask is whether this event represents a "bubble" or a return to fundamentals. Pastor and Veronesi (2003, 2006; henceforth P&V) review the bubble literature, and argue in favor of the hypothesis that technology stocks were overpriced relative to fundamentals. They develop a highly sophisticated version of the Gordon model in continuous time (there are 20 parameters), and analyze the role of uncertainty about the long-run dividend growth rate. In this section we show that in a much simpler framework, the CAPM leads to similar results provided one accounts for mean reversion in profitability.

Firms are assumed to follow a constant payout-ratio dividend policy. To capture the smooth behavior of dividends, let $c$ be the constant proportion of time-$t$ book equity ($B_t$) paid out as a periodic dividend: $D_{t+1}=cB_t$. To model the large number of firms that pay no dividends one may set $c=0$. Define also $\rho_{t+1}$ as the accounting rate of return on book equity (ROE): firm's earnings – as of end of period $t+1$, divided by book value of equity as of period $t$. Then, because of the clean surplus accounting relation, book equity value increases with earnings less dividends paid:†

$$B_{t+1} = (e^{\rho_{t+1}-c})B_t. \tag{12}$$

To model the (continuous time) profitability rate, P&V assume a first-order autoregressive process: $\rho_{t+1} = (1-\phi)\tilde{\rho}_t + \phi\,\rho_t + \varepsilon_{t+1}$, where $\tilde{\rho}_t$ represents long-run profitability. However, the long-run rate itself may be mean reverting as a result of the economy-wide business cycle. Thus, the reduced form model for the book ROE is analogous to an ARMA(2,1) process:

$$\rho_t = (1-\phi_1-\phi_2)\bar{\rho} + \phi_1\rho_{t-1} + \phi_2\rho_{t-2} + \varepsilon_t - \theta_1\varepsilon_{t-1}, \tag{13}$$

where $\bar{\rho}$ is the long-run mean profitability rate. We assume that profitability rate shocks follow the pattern of white noise with variance $\sigma_\varepsilon^2$, and have constant covariance with the market return. This covariance – divided by the variance of the market return, is defined as the profitability rate beta: $\beta_\rho$. The last assumption needed to complete the model is that at a future date $T$ competition will reduce abnormal returns to the point where market value equals book value: $M_T = B_T$.‡ If $\bar{\rho}$ is known with certainty, proposition 4 shows how to obtain the current market-to-book (M/B) ratio.

**Proposition 4:** *Suppose the CAPM holds, and the time series behavior of the profitability rate follows an* $ARMA(2,1)$ *model with a known long-run profitability rate* $\bar{\rho}$. *Define the autocovariance auxiliary variables* $z_j = 1 + \phi_1 z_{j-1} + \phi_2 z_{j-2}$ *and* $w_j = z_j - \theta_1 z_{j-1}$, *for* $j=1,\dots,T$ *and with starting values* $z_0 = z_{-1} = 0$. *Then, the market-to-book ratio is given by*

$$\frac{M_t}{B_t} = \frac{c}{1+R_\mathrm{f}} + c\sum_{\tau=1}^{T-1}\mathrm{H1}(\bar{\rho})_\tau\Gamma_\tau + \mathrm{H1}(\bar{\rho})_T\Gamma_T, \tag{14}$$

*where*

$$\mathrm{H1}(\bar{\rho})_\tau = e^{\bar{\rho}(1-\phi_1-\phi_2)\sum_{j=1}^{\tau} z_j + (\phi_1\rho_t + \phi_2\rho_{t-1} - \theta_1\varepsilon_t)z_\tau + \phi_2\rho_t z_{\tau-1} - \tau c + (\sigma_\varepsilon^2/2)\sum_{j=1}^{\tau} w_j^2},$$

*and*

$$\Gamma_\tau = \prod_{j=1}^{\tau}\frac{1-(ER_\mathrm{m}-R_\mathrm{f})w_j\beta_\rho}{1+R_\mathrm{f}}.$$

Proposition 4 shows that the CAPM leads to a fairly straightforward relationship between the market-to-book ratio and the profitability rate parameters. For example, M/B is positively related to the current rate $\rho_t$, the long-run rate $\bar{\rho}$, and the volatility of profits. We note also that an increase in the risk-free rate, the market risk premium, or profitability rate beta lowers the market-to-book ratio.

†Equation (12) holds exactly only as the length of one period approaches 0. Specifically, the rate of growth in book equity value is $B_{t+1} = (e^{\ln(1+\rho_{t+1}-c)})B_t \approx (e^{\rho_{t+1}-c})B_t$, which becomes exact in continuous time.

‡Pastor and Veronesi (2003) discuss this assumption in detail, and allow for a stochastic horizon in P&V (2006). To simplify the presentation, we keep the time horizon parameter $T$ as a fixed parameter.

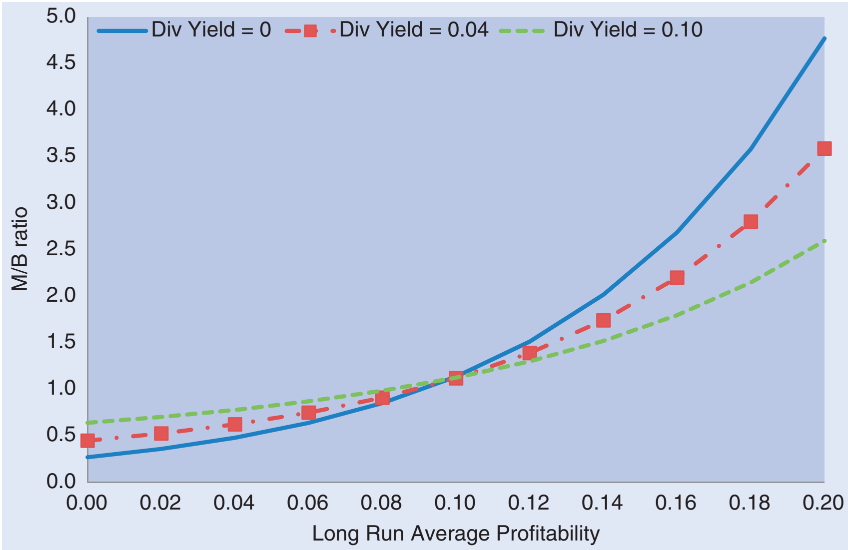


Figure 1. Relationship between M/B ratio and long-run average profitability. The vertical axis represents the current market-to-book (M/B) ratio assuming that the long-run average profitability ($\bar{\rho}$) is known with certainty, for different levels of dividend yield. The model parameter values are: $T=15$, $\phi_1=0.397$ and $\phi_2=0.0$, the current rate $\rho_t=0.11$, $\sigma_\varepsilon^2=0.0834^2$, and $\beta_\rho=0.85$. $R_f=0.03$, and the market risk premium is set to 0.051.



In sum, the results obtained by P&V (2003) with a stochastic discount factor also hold within the context of the CAPM in a simplified framework.

It is easy to see from equation (14) that M/B is a convex function of $\bar{\rho}$. Figure 1 illustrates this relationship for three levels of the payout ratio: $c=0.0$, 0.04, and 0.10. The other parameters are chosen to be as close as possible to those obtained by P&V (2003) from a large sample of firms from CRSP/Compustat database over the time period from 1962 through 2000. Thus, we set $T=15$, $\phi_1=0.397$ and $\phi_2=0.0$, the current rate $\rho_t=0.11$, the idiosyncratic variance $\sigma_\varepsilon^2=0.0834^2$, and the profitability rate beta $\beta_\rho=0.85$. We set the risk-free rate at 0.03, and the market risk premium at 0.051.

From figure 1, the following patterns are evident: First, M/B is convex in long-run profitability. Second, this convexity increases with $\bar{\rho}$ but decreases as the payout ratio $c$ increases. Third, and last, a higher level of $c$ increases M/B when the long-run rate is low because the dividends are received earlier. Alternatively, for highly profitable firms, an increase in the dividend rate leads to a lower market-to-book ratio. Figure 1 confirms the intuition from Corollaries 1 and 2 in P&V (2003). But, the big surprise is that our results follow from a recursive application of the CAPM; moreover, learning about profitability may be easily incorporated into our model as we show next.

### 3.3. Stock valuation with an unknown profitability rate

In this section, we assume that the ROE parameter $\bar{\rho}$ is unknown. Let $E_0\bar{\rho}$ and $V_0\bar{\rho}$ represent the prior mean and variance of investors' beliefs about long-run average profitability. Investors consider the pair of random variables $(\bar{\rho}, \rho_t)$ and learn about $\bar{\rho}$ by observing the current rate $\rho_t$. The following proposition applies Bayes rule to derive the posterior distribution of long-run ROE after observing the full sample $(\rho_1, \rho_2, \ldots, \rho_t)$.

**Proposition 5:** *Suppose at time $t=0$ investors' opinions about long-run profitability are normally distributed: $\bar{\rho}\sim N(E_0\bar{\rho},\ V_0\bar{\rho})$. The distribution of $\bar{\rho}$, after observing the sample $(\rho_1, \rho_2, \ldots, \rho_t)$, is also normal with posterior mean*

$$E_t\bar{\rho} = (1-k_t)(E_0\bar{\rho}) + k_t\hat{\bar{\rho}}_t/(1-\phi_1-\phi_2), \tag{15A}$$

*and variance*

$$V_t\bar{\rho} = (1-k_t)(V_0\bar{\rho}), \tag{15B}$$

*where*

$$k_\tau = \frac{V_0\bar{\rho}}{\sigma_\varepsilon^2}(1-\phi_1-\phi_2)^2\sum_{j=1}^{t} x_j^2, \quad \hat{\bar{\rho}}_t = \frac{\sum_{j=1}^{t} x_j y_j}{\sum_{j=1}^{t} x_j^2},$$

*and the $(x, y)$ variables are computed recursively as $x_j=\theta_1 x_{j-1}+1$, and $y_j=\theta_1 y_{j-1}+\rho_j-\phi_1\rho_{j-1}-\phi_2\rho_{j-2}$ for $j=1,\ldots,t$ with starting values $x_0=y_0=0$.*

Given this description of the learning process, proposition 6 shows how to compute the M/B under the assumption that long-run average profitability is unknown. It will become transparent that higher uncertainty about $\bar{\rho}$ leads to higher market-to-book values.

**Proposition 6:** *Suppose the long-run profitability rate $\bar{\rho}$ is unknown, and investors revise their beliefs according to proposition 5. Then, the market-to-book ratio is given by*

$$\frac{M_t}{B_t} = \frac{c}{1+R_f} + c\sum_{\tau=1}^{T-1}\mathrm{H2}(E_t\bar{\rho},\ V_t\bar{\rho})_\tau\Gamma_\tau + \mathrm{H2}(E_t\bar{\rho},\ V_t\bar{\rho})_T\Gamma_T, \tag{16}$$

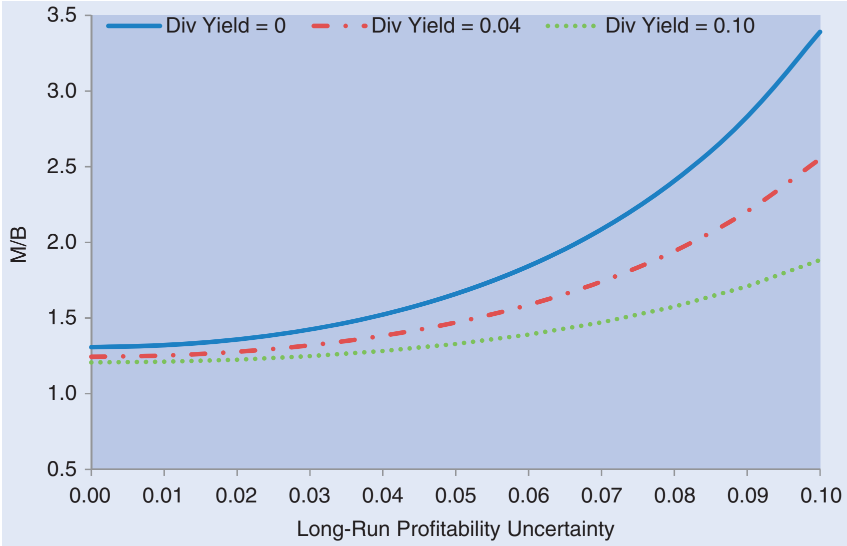


Figure 2. Relationship between M/B ratio and profitability uncertainty. The vertical axis represents the current market-to-book (M/B) ratio as a function of the long-run average profitability uncertainty ($\sqrt{V_t\bar{\rho}}$), for different levels of dividend yield. The model parameter values are: $T=15$, $\phi_1=0.55$, $\phi_2=0.0$ and $\theta_1=0.33$; the remaining parameters are the same as in figure 1.



*where* $\mathrm{H2}(E_t\bar{\rho}, V_t\bar{\rho})_\tau = e^{[(1-\phi_1-\phi_2)\sum_{j=1}^{\tau} z_j]^2 (V_t\bar{\rho}/2)} \mathrm{H1}(E_t\bar{\rho})_\tau$, *and the discount factor* $\Gamma_\tau$ *was defined in proposition 4.*

It is easy to see from proposition 6 that the market-to-book ratio is a convex function of $V_t\bar{\rho}$. Thus, higher uncertainty about long-run profitability leads to a higher M/B ratio. Intuitively, this is so because a sequence of highly profitable years has a greater impact on expected future book value than a similar run of low growth. In turn, a higher future book value results in a higher current market-to-book ratio.

This effect may be seen in figure 2, especially so for firms that pay no dividends. This figure presents the M/B ratio as a function of the posterior profitability uncertainty ($\sqrt{V_t\bar{\rho}}$). The values are computed with $\phi_1=0.55$, $\phi_2=0.0$, and $\theta_1=0.33$. The current rate $\rho_t=0.11$ and the remaining parameters are the same as in figure 1. What is most striking is how close our results are to those in the top panel of figure 3 of P&V. Once again, proposition 6 and figure 2 confirm that, properly applied, the CAPM has the potential to explain the observed impact of learning on stock valuation.

### 3.4. *The St. Petersburg paradox*

The similarity between the St. Petersburg game and a growth stock valuation model was first recognized by Durand (1957). Suppose that the firm's expected future dividends (as the game's future payoffs) grow at a constant rate; if dividends are discounted at a constant rate, then to obtain a finite present value this discount rate must be greater than the dividend growth rate. The value of the stock is infinite if dividends grow at an equal or higher rate than the discount rate; and in the same way, the expected payoff of the St. Petersburg game is infinite only when the payoffs are increasing at an equal or higher rate than the rate at which the correspondent probabilities are decreasing.

In light of the results in Section 2.1, both assumptions appear to be unrealistic. The dividend stream is unlikely to grow at a constant rate in perpetuity because competition will eliminate abnormal earnings after a period of time. The second assumption of a constant cost of capital is also unnecessary because proposition 1 shows how to set the risk-adjusted discount rate.

To derive the condition under which the asset value is finite, observe that for large $j$, $z_j$ converges to a constant value of $1/(1-\phi_1)$. Therefore, the expected future dividend will evolve along the path $A_T = e^{(\bar{g}+(1/2)[\sigma_\varepsilon^2/(1-\phi_1)^2])T}$. Also, for dividends far into the future, the risk-adjusted present value factor may be approximated by $[\{1-(ER_{\mathrm{m}}-R_{\mathrm{f}})\beta_{\mathrm{g}}/(1-\phi_1)\}/(1+R_{\mathrm{f}})]^T$. Therefore, the present value of $D_T$, for large $T$, may be approximated as follows:

$$V_0 \approx D_0\, e^{[\bar{g}+(1/2)[\sigma_\varepsilon^2/(1-\phi_1)^2]-R_{\mathrm{f}}-(ER_{\mathrm{m}}-R_{\mathrm{f}})[\beta_g/(1-\phi_1)]]T} \qquad (17)$$

This value will converge to zero, and thereby the sum of the present values of all future dividends (i.e. the firm value) will be finite, provided the expression inside the square brackets is negative. Thus, the restriction on long term growth is

$$\bar{g} + \frac{1}{2}\frac{\sigma_\varepsilon^2}{(1-\phi_1)^2} \leq R_{\mathrm{f}} + (ER_{\mathrm{m}} - R_{\mathrm{f}})\frac{\beta_{\mathrm{g}}}{(1-\phi_1)}. \qquad (18)$$

The right-hand side of this expression is similar to the CAPM risk-adjusted return. The only difference is that risk is measured by the growth rate beta adjusted by the degree of predictability in the earnings stream. The left-hand side consists of the long-run growth rate adjusted by one-half the long-run variance of earnings growth. This condition is less restrictive than the one for the constant growth (i.e. $g < r$). First, the condition allows a very high growth rate of dividends for a certain number of years (it depends only on the long-run growth rate); and second, since the risk-premium is positive, the discount rate is higher than the risk-free rate.

## 4. Conclusions

The main contributions of this paper may be summarized as follows. Suppose the dividend growth rate is mean reverting, then the CAPM implies that rate of return beta must vary with the strength of mean reversion. We derive explicit formulas for the return beta, return volatility, and Sharpe ratio assuming dividend growth follows an autoregressive moving-average process. When the model is calibrated to display even a small degree of negative correlation – consistent with the empirical results in L&W, the rate of return beta, and the return volatility fall as the time horizon increases. Thus, firms with low duration cash flows exhibit higher systematic risk than firms with long duration cash flows; and this is all required theoretically in order to explain the well known value premium anomaly.

A second application of our methodology deals with stock valuation when long-run profitability is unknown. We model the accounting rate of return on equity with an ARMA process; however, investors do not know how profitable a firm might be in the long-run. They rationally learn about its potential value by observing the current rate. We prove that within a CAPM world the market-to-book ratio is a convex function of the uncertainty related to long-run profitability. Thus, young firms may be more valuable because relatively little is known about their long-run potential.

The last application of our model is to the classical St. Petersburg paradox. To preclude an infinite present value, a stock with a constant dividend growth rate requires that the discount rate must be greater than the rate of growth. We argue that the current dividend growth rate can be very large but the long-run rate cannot be that far away from the growth rate of the macro economy. We show that the CAPM implies a non-constant risk-adjusted discount rate. These two observations lead to a new, much less stringent restriction on the growth rate that is independent of the short-run growth rate, and a new potential resolution of the paradox.

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## Appendix A

To apply the dividend discount model we need to forecast the path of future expected dividends given that the growth rate follows a $p$th-order autoregressive process AR($p$):

$$g_{t+1} = (1 - \phi_1 \cdots - \phi_p)\bar{g} + \phi_1 g_t \cdots + \phi_p g_{t-p+1} + \varepsilon_{t+1}, \tag{A.1}$$

where $\bar{g}$ is the long-run (unconditional) mean growth rate and $\phi_1, \ldots, \phi_p$ are the autoregressive coefficients. Then, the CAPM may be applied recursively (in the fashion of dynamic programming) to account for market risk.

The purpose of this Appendix is to derive the conditional expectation of a future dividend $D_T = D_0\, e^{\sum_{t=1}^{T} g_t}$. Observe that, conditional on $p$ previous growth rates $(g_0, \ldots, g_{1-p})$, this expectation is the moment generating function of the cumulative growth evaluated at the real value 1. Given the assumptions on $\varepsilon_{t+1}$, it follows that expected dividends depend only on the mean and variance of $\sum_{t=1}^{T} g_t$.

To obtain a simple closed-form solution, observe that the sequence of future growth rates $G' \equiv (g_1, g_2, \ldots, g_T)$ has the following representation in matrix form: $\Phi G = (1 - \phi_1 \cdots - \phi_p)\bar{g}i + \Phi_0 G_0 + \mathrm{E}$ where the $T \times T$ semi-difference matrix $\Phi$ is constructed as follows: the first row consists of 1 followed by $T-1$ 0 s, the second one consists of $-\phi_1$, followed by 1 and $T-2$ 0 s. Similarly, the third row is $-\phi_2, -\phi_1, 1$ and $T-3$ 0 s, on down to the last row which consists of $T-1-p$ 0 s followed by $-\phi_p, \ldots, -\phi_1, 1$. For the $T \times p$ matrix $\Phi_0$ the first row consists of $\phi_1, \ldots, \phi_p$, followed by $0, \phi_2, \ldots, \phi_p$ for the second row, down to the $p$th row which is $0, \ldots, 0, \phi_p$. The remaining rows are all 0 s. $i$ is a column vector of 1 s and the initial growth rates are $G_0' \equiv (g_0, g_{-1}, \ldots, g_{1-p})$. $E' \equiv (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_T)$ is a vector of random innovation terms. Using this setup, the cumulative growth rate has conditional mean $(1 - \phi_1 \cdots - \phi_p)\bar{g}\, i'\Phi^{-1}i + i'\Phi^{-1}\Phi_0 G_0$ and variance $\sigma_\varepsilon^2\, i'\Phi^{-1}(\Phi^{-1})'i$. Using a result from Ali (1977), we show next that these moments may be computed without the need to invert the $\Phi$ matrix. Define the vector $Z' \equiv (z_T, z_{T-1}, \ldots, z_1) = i'\Phi^{-1}$ and note that each element may be computed recursively from the previous one: $z_j = \phi_1 z_{j-1} + \cdots + \phi_p z_{j-p} + 1$ for $j = 1, 2, \ldots, T$, and starting values of $z_j = 0$ for $j = 0, -1, \ldots, 1-p$. Hence, the expected time-$T$ dividend may be computed as

$$E_0 D_T = D_0\, A_T e^{z_T(\phi_1 g_0 + \cdots + \phi_p g_{1-p}) + \cdots + z_{T-p+1}(\phi_p g_{1-p})}, \tag{A.2}$$

where $A_T = e^{(1-\phi_1 \cdots - \phi_p)\bar{g} \sum_{j=1}^T z_j + (\sigma_\varepsilon^2/2)\sum_{j=1}^T z_j^2}$.

## Appendix B: Proofs

This Appendix contains the proofs of propositions 2 thru 6. Proposition 1 in the paper is a special case of 1A below.

**Proposition 1A:** *Suppose the dividend growth rate follows a pth-order autoregressive process as in equation* (A.1). *Define the auxiliary variable $z_j = \phi_1 z_{j-1} + \cdots + \phi_p z_{j-p} + 1$ for $j = 1, 2, \ldots, T$, and starting values of $z_j = 0$ for $j = 0, -1, \ldots, 1-p$. Then, the present value of a single future dividend $D_T$ is given by its conditional expected future value adjusted for its market risk and discounted to the present at the risk-free rate of interest:*

$$V_0 = \frac{(E_0\, D_T)\prod_{j=1}^T [1 - (ER_\mathrm{m} - R_\mathrm{f}) z_j \beta_\mathrm{g}]}{(1 + R_\mathrm{f})^T}, \tag{B.1}$$

*where the growth rate beta, $\beta_g$, is defined as the covariance between the growth rate innovation and the market return, divided by the variance of the market return. Summing over all future expected dividends yields the CAPM price for an asset with a stochastic growth rate:*

$$P_0 = \sum_{t=1}^{\infty} \frac{(E_0\, D_t)\prod_{j=1}^t [1 - (ER_\mathrm{m} - R_\mathrm{f}) z_j \beta_\mathrm{g}]}{(1 + R_\mathrm{f})^t}. \tag{B.2}$$

**Proof:** The proof is by induction on $t$. Clearly, at time $T$, $V_T = D_T$. Let $T$–1 be one period prior to the realization of the earnings or dividend; insert the return $R_T = (D_T/V_{T-1}) - 1$ into the security market line (equation (3)) to show that the discounted value of $D_T$ is given by

$$V_{T-1} = E_{T-1} D_T \times \left[\frac{1 - (ER_\mathrm{m} - R_\mathrm{f})\mathrm{Cov}(D_T/(E_{T-1}D_T), R_{\mathrm{m}T})/\sigma_\mathrm{m}^2}{1 + R_\mathrm{f}}\right].$$

Using Stein's lemma it follows that

$$\mathrm{Cov}\left(\frac{D_T}{E_{T-1}\, D_T}, R_{\mathrm{m}T}\right) = \mathrm{Cov}(\varepsilon_T, R_{\mathrm{m}T}).$$

Next, define the growth rate beta as $\beta_\mathrm{g} = \mathrm{Cov}(\varepsilon_T, R_{\mathrm{m}\,T})/\sigma_\varepsilon^2$, and substitute this beta into the valuation formula to show that equation (4A) holds for $t = T-1$. Assume the result holds for time period $t = \tau + 1$. Then, the ratio of $V_{\tau+1}$ to its conditional expectation as of one prior period, $E_\tau V_{\tau+1}$, equals the ratio of recursive cash flow expectations:

$$\frac{V_{\tau+1}}{E_\tau\, V_{\tau+1}} = \frac{E_{\tau+1}\, D_T}{E_\tau\, D_T}.$$

Once again, we use Stein's lemma to show that the

$$\mathrm{Cov}\left(\frac{V_{\tau+1}}{E_\tau\, V_{\tau+1}}, R_{\mathrm{m},\,\tau+1}\right) = \mathrm{Cov}\left(\sum_{s=\tau+1}^{T} g_s, R_{\mathrm{m},\,\tau+1}\right).$$

To simplify the last expression, observe that the stochastic component of the aggregate growth rate is $i'\Phi^{-1}\mathrm{E}$. Hence, the covariance has a simple closed form solution: $z_{T-\tau}\,\mathrm{Cov}(\varepsilon_{\tau+1}, R_{\mathrm{m},\,\tau+1})$, and the discount factor from period $\tau+1$ to $\tau$ becomes

$$\left[\frac{1 - (ER_\mathrm{m} - R_\mathrm{f}) z_{T-\tau}\beta_\mathrm{g}}{1 + R_\mathrm{f}}\right].$$

This last step shows that equation (B.1) holds for all time periods including $t = 0$. The second part of the proposition (B.2) holds by the principle of value additivity. □

**Proof of proposition 2:** The core of the proof is based on the sequence of growth rates $G' \equiv (g_{t+1}, \ldots, g_{t+T})$. The entire sample has the following representation in matrix form:

$$\Phi G = (1 - \phi)\bar{g}i + \Theta \mathrm{E} + G_0, \tag{B.3}$$

where the $T \times T$ matrix $\Phi$ consists of 1 s along the main diagonal, $-\phi_1$ in each cell right below the main diagonal and 0s everywhere else. $\Theta$ has the same dimensions and is defined similarly but with $\theta_1$ in place of $\phi_1$; $i$ is column vector of 1s, and $G_0$ is a column vector with initial condition $\phi_1 g_t - \theta_1 \varepsilon_t$ in the first row, and 0 s in the remaining rows. $E' \equiv (\varepsilon_{t+1}, \varepsilon_{t+2}, \ldots, \varepsilon_{t+T})$ is a row vector of growth rate shocks. It follows immediately that the conditional mean and variance of the cumulative growth rate $\sum_{s=1}^T g_{t+s}$ are, respectively: $E_t\, i'G = (1 - \phi) \times \bar{g}i'\Phi^{-1}i + i'\Phi^{-1}G_0$ and $V_t\, i'G = \sigma_\varepsilon^2 (i'\Phi^{-1}\Theta)(i'\Phi^{-1}\Theta)'$.

These expressions may be simplified considerably. To this end, define the vector $Z' \equiv (z_T, z_{T-1}, \ldots, z_1) = i'\Phi^{-1}$ so that each element may be computed recursively from the previous one: $z_j = \phi_1 z_{j-1} + 1$, and starting value of $z_0 = 0$. Define also the vector $W' \equiv (w_T, w_{T-1}, \ldots, w_1) = Z'\Theta$ to aggregate serial correlation induced by the moving average component of growth. Each element may be computed recursively as a follows: $w_j = z_j - \theta_1 z_{j-1}$ for $j = 1, 2, \ldots, T$. Given these transformation, the conditional expected future cash flow (equation (10B)) follows immediately.

The rest of the proof is by induction on $t$. From proposition 1, we know that at $T-1$ the discounted value of $D_T$ is given by

$$V_{T-1} = E_{T-1}\, D_T\left[\frac{1-(ER_{\mathrm{m}} - R_{\mathrm{f}})\beta_{\mathrm{g}}}{1+R_{\mathrm{f}}}\right].$$

Thus, equation (10A) holds as of $T-1$ because the first value of $w$ is 1. Assume the result holds for time period $t = \tau + 1$. From Stein's lemma we have

$$\begin{aligned}\mathrm{Cov}_\tau\left(\sum_{s=\tau+1}^{T} g_s, R_{\mathrm{m},\tau+1}\right) &= \mathrm{Cov}_\tau(W'E, R_{\mathrm{m},\tau+1})\\ &= w_{T-\tau}\mathrm{Cov}_\tau(\varepsilon_{\tau+1}, R_{\mathrm{m},\tau+1}).\end{aligned}$$

Using the same logic as in proposition 1, as we move back one time period from $\tau+1$ to $\tau$, the discount factor is

$$\left[\frac{1-(ER_{\mathrm{m}} - R_{\mathrm{f}})w_\tau\beta_{\mathrm{g}}}{1+R_{\mathrm{f}}}\right].$$

Thus, the time $t = T - \tau$ price is given by

$$V_{t,T} = \frac{(E_{T-\tau}\, D_T)\prod_{j=1}^{\tau}[1-(ER_{\mathrm{m}} - R_{\mathrm{f}})w_j\beta_{\mathrm{g}}]}{(1+R_{\mathrm{f}})^\tau}.$$

This last step shows that the proposition holds for time period $t = T - \tau$, and all other times $t$. □

**Proof of proposition 3:** The covariance of the market return with the return's random component has an explicit solution

$$\mathrm{Cov}(e^{w_T\varepsilon_{t+1}}, R_{\mathrm{M},t+1}) = w_T(Ee^{w_T\varepsilon_{t+1}})\mathrm{Cov}(\varepsilon_{t+1}, R_{\mathrm{M},t+1}).$$

Thus, by definition of rate of return beta we have

$$\beta_{\mathrm{ROR},T} = \frac{\mathrm{Cov}(R_{T,t+1}, R_{\mathrm{M},t+1})}{\sigma_{\mathrm{m}}^2} = \frac{(1+R_{\mathrm{f}})w_T\beta_{\mathrm{g}}}{1-(ER_{\mathrm{m}} - R_{\mathrm{f}})w_T\beta_{\mathrm{g}}},$$

and part (11A) holds. To obtain the rate of return variance, note that the random variable $w_T\varepsilon_{t+1}$ is normally distributed with mean 0 and variance $w_T^2\sigma_\varepsilon^2$. Therefore, equation (11B) follows from the properties of a lognormal random variable. Last, we use the security market line (equation (3)), to show that the risk premium is given by

$$ER_{T,t+1} - R_{\mathrm{f}} = \left[\frac{(1+R_{\mathrm{f}})w_T\beta_{\mathrm{g}}}{1-(ER_{\mathrm{m}} - R_{\mathrm{f}})w_T\beta_{\mathrm{g}}}\right](ER_{\mathrm{m}} - R_{\mathrm{f}}),$$

and the Sharpe ratio (equation (11C)) follows by definition. □

**Proof of proposition 4:** The full sample of ROEs $R' \equiv (\rho_{t+1}, \ldots, \rho_{t+T})$ may be represented in matrix form as: $\Phi R = (1-\phi_1-\phi_2)\bar{\rho}i + \Theta \mathrm{E} + R_0$. The square matrix $\Phi$ has T columns; the first consists of 1 in the first row, followed by $-\phi_1$, $-\phi_2$, and $T-3$ 0 s in the remaining rows. The second column has 0 in the first row, followed by 1, $-\phi_1$, $-\phi_2$, and $T-4$ 0 s. The remaining columns have the same format up to column $T$ which consists of $T-1$ 0 s and 1 in the last row. The matrix $\Theta$ has the same dimensions as $\Phi$, and is defined similarly but with two changes: $\theta_1$ in place of $\phi_1$ and 0 in place of $\phi_2$. $i$ is a column vector of 1s, and $R_0$ is a column vector with initial conditions: $\phi_1\rho_t + \phi_2\rho_{t-1} - \theta_1\varepsilon_t$ in the first row, $\phi_2\rho_t$ in the second row, and 0s in the $T-3$ remaining rows. $E' \equiv (\varepsilon_{t+1}, \varepsilon_{t+2}, \ldots, \varepsilon_{t+T})$ is a row vector of profitability shocks. Given this set up, the derivation of equation (14) is analogous to that of proposition 2, hence it is omitted. □

**Proof of proposition 5:** Let $R' \equiv (\rho_1, \ldots, \rho_t)$ be the sample of ROEs from period 1 thru $t$; the matrix representation for $R$ is the same as that described in proposition 4 with the obvious change in sample size. We assume that as of time $t = 0$, investors form beliefs about the distribution of $\bar{\rho}$, and use the ARMA process to set expectations about the joint behavior of $\bar{\rho}$ and the sample $(\rho_1, \rho_2, \ldots, \rho_t)$. Thus, the covariance of long run profitability and $R$ is given by: $\mathrm{Cov}_0(\bar{\rho}, R) = (1-\phi_1-\phi_2)(V_0\bar{\rho})\Phi^{-1}i$. The mean and variance of $R$ are: $E_0R = (1-\phi_1-\phi_2)(E_0\bar{\rho})\Phi^{-1}i + \Phi^{-1}R_0$, and $V_0R = \sigma_\varepsilon^2\,\Phi^{-1}\Theta\,\Theta'(\Phi^{-1})'$. Next, apply the partition theorem for normal random variables to arrive at the posterior moments:

$$\begin{aligned}E_t\bar{\rho} \equiv E(\bar{\rho}|\rho_1, \ldots, \rho_t) &= E_0\bar{\rho}\\ &+ (V_0\bar{\rho})(1-\phi_1-\phi_2)i'(\Phi^{-1})'[\sigma_\varepsilon^2\Phi^{-1}\,\Theta\Theta'(\Phi^{-1})']^{-1}\\ &\times (R - E_0R),\end{aligned}$$

and

$$\begin{aligned}V_t\bar{\rho} \equiv V(\bar{\rho}|\rho_1, \ldots, \rho_t) &= V_0\bar{\rho} - (V_0\bar{\rho})^2\\ &\times (1-\phi_1-\phi_2)^2 i'(\Phi^{-1})'[\sigma_\varepsilon^2\Phi^{-1}\Theta\Theta'(\Phi^{-1})']^{-1}\Phi^{-1}i.\end{aligned}$$

The proof is completed once we set the $(x, y)$ variables. First, define the vector $Y' \equiv (y_1, y_2, \ldots, y_t) = \Theta^{-1}\mathrm{P}'$, where the $j$th element of the column vector $\mathrm{P}'$ is $\rho_j - \phi_1\rho_{j-1} - \phi_2\rho_{j-2}$ for $j = 1$ thru $t$. Thus, $y_j = \theta_1 y_{j-1} + (\rho_j - \phi_1\rho_{j-1} - \phi_2\rho_{j-2})$, with a starting value of $y_0 = 0$. Second, define the vector $X' \equiv (x_1, x_2, \ldots, x_t) = \Theta^{-1}1$ to aggregate serial correlation induced by the moving average component of profitability. Each element may be computed recursively as a follows: $x_j = 1 + \theta_1 x_{j-1}$ for $j = 1, 2, \ldots, t$. □

**Proof of proposition 6:** Proposition 5 shows that the posterior distribution of $\bar{\rho}$ is normal with mean and variance given by equations (15A) and (15B). Therefore, the distribution of $\bar{\rho}(1-\phi_1-\phi_2)\sum_{j=1}^{t} z_j$ is also normal with mean $[(1-\phi_1-\phi_2)\sum_{j=1}^{t} z_j]E_t\bar{\rho}$ and variance $[(1-\phi_1-\phi_2)\sum_{j=1}^{t} z_j]^2 V_t\bar{\rho}$. The expectation of $\mathrm{H1}(\bar{\rho})$ over the posterior distribution of $\bar{\rho}$ yields $e^{[(1-\phi_1-\phi_2)\sum_{j=1}^{t} z_j]^2(V_t\bar{\rho}/2)}$ times H1 evaluated at $E_t\bar{\rho}$ in place of $\bar{\rho}$. □